

Commuting Squares and Index for Subfactors

(Ph.D Thesis)

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Summary

The index for subfactors was introduced by V. Jones in [Jo]. In this paper he proves that the only values of the index for a subfactor, A , of the hyperfinite II_1 -factor, R , in the interval $[1, 4)$ are the numbers

$$\{4 \cos^2 \frac{\pi}{n} \mid n = 3, 4, \dots\}.$$

The values above 4 of the index for irreducible subfactors of R are not known today, but recently S. Popa [Po1] proved, that all real numbers above 4 can be the index of a irreducible pair of non-hyperfinite factors.

It seems likely, that the values of the index for irreducible subfactors of R , are related to finite or infinite graphs, in the sense that the index values has to be the square of the norm of such graphs [Po2]. The close connection with graphs is also reflected in the present work, where all the given examples of the index, arise as the square of the norm of a finite or infinite graph.

The first real breakthrough in constructing values of the index for irreducible subfactors of R , came with H. Wenzl's work [Wen2], where he uses periodic ladders of inclusions of multi-matrix algebras to construct subfactors of R . He also gives an easy way to determine the index for the constructed pair, and a very useful criterion for determining irreducibility of the pair. Wenzl constructs the following values of the index

$$\frac{\sin^2 \frac{k\pi}{l}}{\sin^2 \frac{\pi}{l}}, \quad l \geq 3, \quad 2 \leq k \leq l-2.$$

A key ingredient in building periodic ladders of multi-matrix algebras, is the notion of a commuting square (see [G.H.J.] chapter 4), which consist of four multi-matrix algebras A, B, C and D , included in each other via inclusion matrices G, H, K and L

$$\begin{array}{ccc} C & \subset_L & D \\ \cup_K & & \cup_H \\ A & \subset_G & B \end{array}$$

together with a faithful trace tr_D on D , such that $E_A = E_B E_C = E_C E_B$, where E_X denotes the unique trace preserving conditional expectation of D onto X , $X = A, B, C$.

The present work is divided into 6 chapters.¹ In the beginning of chapter 1 we give a characterization of commuting squares of multi-matrix algebras, the *bi-unitary condition*, which also occurs in the unpublished work of A. Ocneanu, in a slightly different setting. If the inclusion matrices satisfy $HL^t = G^t K$ in addition to the necessary condition $GH = KL$, the above mentioned ladder construction will, under certain mild extra assumptions on G, H, K and L , produce an irreducible subfactor of the Hyperfinite II_1 -factor of index $\|H\|^2 = \|K\|^2$. The main part of chapter 1 is used to determine which matrices, G , can be used to build a commuting square of the form

$$\begin{array}{ccc} C & \subset_{nG^t} & D \\ \cup_G & & \cup_{G^t} \\ A & \subset_{nG} & B \end{array} \quad n \in \mathbb{N} \quad (i)$$

¹Chapter 1 and chapter 2 are joint work with Prof. Uffe Haagerup, Dept. of Mathematics and Computer Science, Odense University.

under the assumption that the resulting index values, $\|G\|^2$, should be in the interval $(4, 5)$. The lowest value obtained this way is

$$\frac{5 + \sqrt{13}}{2} \approx 4.302$$

Following a suggestion by A. Ocneanu [O], we, in chapter 2, obtain index values closer to 4, by considering commuting squares of the form

$$\begin{array}{ccc} C & \subset_{G^t G - I} & D \\ \cup_G & & \cup_G \\ A & \subset_{G G^t - I} & B \end{array} \quad (\text{ii})$$

Moreover we determine which matrices, G , that can occur in (ii), under the assumption that the associated graph Γ_G , has the form of a star with three rays.

In chapter 2 we also bring a presentation of a construction of a commuting square based on the graph E_{10} . The example was originally conceived by A. Ocneanu [O], and is particularly interesting, since the index value obtained from this graph, is the lowest (above 4) which can be obtained as $\|G\|^2$, for G a matrix with non-negative integer entries.

In chapter 3 we bring a construction of commuting squares, based on the Dynkin diagrams A_l . The index values constructed this way are

$$\frac{\sin^2 \frac{k\pi}{l}}{\sin^2 \frac{\pi}{l}}, \quad l \geq 3, \quad 2 \leq k \leq l - 2,$$

i.e. the same as Wenzl constructed in [Wen2].

In chapter 4 we define the notion of an infinite dimensional multi-matrix algebra, and show that the theory for building subfactors of R , from commuting squares of multi-matrix algebras, can be generalized to commuting squares of infinite dimensional multi-matrix algebras. We also prove that Wenzl's irreducibility criterion is still valid in this setting.

In chapter 5 we look at some infinite graphs defined by J. Shearer [Sh]. For each $\lambda > \sqrt{2 + \sqrt{5}}$, Shearer defines an infinite graph, Γ_λ , with largest eigenvalue λ . We show that the Perron–Frobenius vector of these graphs is summable, which implies that these graphs might be used to define the inclusions H and K of a commuting square of infinite dimensional multi-matrix algebras. For the only obvious choice of the other inclusions, G and L , it is shown that a commuting square of this form implies that Γ_λ is eventually periodic. This is not a property of Γ_λ in general, so there is no simple way to build a commuting square of infinite dimensional multi-matrix algebras from Γ_λ for general λ .

In chapter 6 we look at another class of infinite graphs, $T(1, n, \infty)$, defined by A. Hoffmann in [Hof]. We show, that the value of the index, which might be constructed using this graph, is not obtainable from a construction on any finite graph, and we construct commuting squares of infinite dimensional multi-matrix algebras based on $T(1, 2, \infty)$, $T(1, 3, \infty)$ and $T(1, 4, \infty)$. Unfortunately there is no general pattern in these constructions, which could show us how to do the construction for general n .

Sammenfatning (Danish Summary)

Index for delfaktorer blev introduceret af V. Jones i [Jo]. I denne artikel bevises det, at de eneste værdier af indexet for en irreducibel delfaktor af R , den Hyperendelige II_1 -faktor, i intervallet $[1, 4)$ er tallene

$$\{4 \cos^2 \frac{\pi}{n} \mid n = 3, 4, \dots\}.$$

Mængden af værdier over 4, som index for en irreducibel delfaktor af den Hyperendelige II_1 -faktor kan antage, er til dato ukendt, men for nylig viste S. Popa [Po1], at alle reelle tal over 4 kan antages af index for par af ikke Hyperendelige irreducible faktorer.

Det virker sandsynligt, at der er en snæver sammenhæng mellem værdierne af index for par af irreducible Hyperendelige II_1 -faktorer og endelige/uendelige grafer, forstået således, at indexværdierne er kvadratet på normer af adjacensmatricer for sådanne grafer [Po2]. Denne sammenhæng er også synlig i dette arbejde, idet alle de konstruerede værdier af index kommer til verden som kvadratet på normen af en endelig eller uendelig graf.

Det første gennembrud i konstruktionen af index for irreducible delfaktorer af R kom med H. Wenzls artikel [Wen2]. Her benyttes periodiske "ladders" af inklusioner af multi-matrix algebraer til at konstruere irreducible delfaktorer af R . Der bestemmes ligeledes en nem måde hvorpå index for de konstruerede delfaktorer kan beregnes, samt et yderst anvendeligt kriterium til at bevise irreducibilitet af af de konstruerede delfaktorer. Wenzl bestemmer følgende værdier af index

$$\frac{\sin^2 \frac{k\pi}{l}}{\sin^2 \frac{\pi}{l}}, \quad l \geq 3, \quad 2 \leq k \leq l-2.$$

Et af de væsentligste redskaber i konstruktionen af periodiske ladders af multi-matrix algebraer er begrebet "commuting squares". Disse består ifølge [G.H.J.] kapitel 4 af fire multi-matrix algebraer A, B, C og D , indlejret i hinanden via inklusionsmatricerne G, H, K og L

$$\begin{array}{ccc} C & \subset_L & D \\ \cup_K & & \cup_H \\ A & \subset_G & B \end{array}$$

samt et tro spor, tr_D , på D , således at $E_A = E_B E_C = E_C E_B$, hvor E_X betegner den entydige "conditional expectation" af D på X , $X = A, B, C$.

Nærværende arbejde er delt i 6 kapitler.² I starten af kapitel 1 gives en karakterisation af commuting squares af multi-matrix algebraer, den såkaldte *bi-unitære betingelse*, der endvidere er en del af A. Ocneanu, endnu ikke offentliggjorte, arbejde, hvor den forekommer i en lidt anden sammenhæng.

Hvis inklusionsmatricerne opfylder $HL^t = G^t K$, ud over den nødvendige betingelse, $GH = KL$, vil den ovenfor nævnte ladder konstruktion, med nogle få, ikke specielt restriktive, antagelser om G, H, K and L , resultere i en irreducibel delfaktor af den Hyperendelige II_1 -faktor med index $\|H\|^2 = \|K\|^2$.

²Kapitel 1 og kapitel 2 er fælles arbejde med Prof. Uffe Haagerup, Institut for Matematik og Datalogi, Odense Universitet.

Hovedparten af kapitel 1 bruges til at bestemme hvilke matricer, G , der kan bruges til konstruktion af commuting squares på formen

$$\begin{array}{ccc} C & \subset_{nG^t} & D \\ \cup_G & & \cup_{G^t} \\ A & \subset_{nG} & B \end{array} \quad n \in \mathbb{N} \quad (\text{iii})$$

hvor det yderligere forudsættes, at den resulterende indexværdi skal tilhøre intervallet $(4, 5)$. Den mindste indexværdi, der fremkommer på denne måde, er

$$\frac{5 + \sqrt{13}}{2} \approx 4.302$$

Idet en ide af A. Ocneanu [O] følges op, konstrueres i kapitel 2 værdier af index, der ligger meget tættere på 4, ved at betragte commuting squares på formen

$$\begin{array}{ccc} C & \subset_{G^t G - I} & D \\ \cup_G & & \cup_G \\ A & \subset_{G G^t - I} & B \end{array} \quad (\text{iv})$$

Desuden bestemmes hvilke matricer, G , der kan benyttes i (iv), under antagelse af, at de korresponderende grafer, Γ_G , har form som en stjerne med tre stråler.

I kapitel 2 bringes ligeledes en præsentation af en konstruktion, der oprindeligt skyldes A. Ocneanu [O]. Denne konstruktion er specielt interessant, idet den resulterer i den mindst mulige værdi af index (over 4), der kan fremkomme som $\|G\|^2$ for en matrix, G , med ikke-negative heltallige koefficienter.

I kapitel 3 bringes en konstruktion af commuting squares baseret på Dynkin diagrammerne A_l . Indexværdierne konstrueret på denne måde er

$$\frac{\sin^2 \frac{k\pi}{l}}{\sin^2 \frac{\pi}{l}}, \quad l \geq 3, \quad 2 \leq k \leq l-2,$$

i.e. de samme værdier som Wenzl konstruerede i [Wen2].

I kapitel 4 defineres begrebet en uendeligdimensional multi-matrix algebra, og det vises at teorien for at konstruere delfaktorer af R , på grundlag af commuting squares af multi-matrix algebraer, kan generaliseres til at gælde for commuting squares af uendeligdimensionale multi-matrix algebraer. Endvidere vises det, at Wenzls kriterium til at bevise irreducibilitet af det konstruerede par af Hyperendelige II_1 -faktorer, også gælder i dette tilfælde.

I kapitel 5 betragtes nogle uendelige grafer, defineret af J. Shearer [Sh]. For ethvert $\lambda > \sqrt{2 + \sqrt{5}}$ definerer Shearer en uendelig graf, Γ_λ , med største egenværdi λ . Her bevises det, at den tilhørende positive egenvektor er summabel, hvilket betyder, at disse grafer måske kan benyttes til at definere inklusionerne H og K i en commuting square af uendeligdimensionale multi-matrix algebraer. For det eneste oplagte valg af inklusioner G og L vises det, at eksistensen af en sådan commuting square implicerer, at grafen Γ_λ er periodisk fra et vist trin. En sådan periodicitet er ikke en egenskab som

Γ_λ generelt er i besiddelse af, så der er ikke nogen nem måde at konstruere en commuting square af uendeligdimensionale multi-matrix algebraer ud fra Γ_λ for et generelt λ .

I kapitel 6 betragtes en klasse af uendelige grafer, $T(1, n, \infty)$, defineret af A. Hoffmann i [Hof]. Det vises, at de værdier af index, der eventuelt kan konstrueres ud fra disse grafer, ikke kan konstrueres på grundlag af nogen endelig graf. Herefter konstrueres eksempler på grundlag af graferne $T(1, 2, \infty)$, $T(1, 3, \infty)$ og $T(1, 4, \infty)$. Desværre afslører der sig ikke noget generelt mønster i disse konstruktioner, der kunne give en ide til en konstruktion for vilkårligt n .

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Part I

Finite Dimensional Commuting Squares

The Simplest Possible Commuting Squares

1 Ocneanu's Bi-unitary Condition and Symmetric Commuting Squares

Following [G.H.J.] chapter IV, a commuting square is a set of four finite von Neumann algebras A , B , C and D , nested in each other by $A \subset B \subset D$ and $A \subset C \subset D$, together with a normal faithful tracial state tr_D on D , such that the unique trace preserving conditional expectations E_A , E_B and E_C of D onto A , B resp. C satisfy

$$E_A = E_B E_C = E_C E_B.$$

If A , B , C and D are multi-matrix algebras, the inclusions $A \subset B \subset D$ and $A \subset C \subset D$ are given by inclusion matrices G , H , K , and L and we will write

$$\begin{array}{ccc} C & \subset_L & D \\ \cup_K & & \cup_H \\ A & \subset_G & B. \end{array}$$

For our purposes we will need the following characterization of a commuting square of multi-matrix algebras.

Lemma 1.1 *Let A , B , C and D be multi-matrix algebras $A \subset B \subset D$, $A \subset C \subset D$ and let tr_D be a trace on D . Set $\langle d_1, d_2 \rangle_{\text{tr}_D} = \text{tr}_D(d_1 d_2^*)$, $d_1, d_2 \in D$. Then the following conditions are equivalent*

1. A , B , C and D form a commuting square with respect to the trace tr_D on D .
2. $A^\perp \cap B$ and $A^\perp \cap C$ are orthogonal with respect to the inner product $\langle \cdot, \cdot \rangle_{\text{tr}_D}$.
3. $\langle E_A(b), E_A(c) \rangle_{\text{tr}_D} = \langle b, c \rangle_{\text{tr}_D}$ for all $b \in B$ and all $c \in C$.

Proof Let tr_A , tr_B and tr_C denote the restriction of tr_D to A , B resp. C .

1 \Rightarrow 2: Let $c \in C$ and $b \in A^\perp \cap B$, then

$$\langle b, c \rangle_{\text{tr}_D} = \text{tr}_C(E_C(bc^*)) = \text{tr}_C(E_C(b)c^*) = \text{tr}_C(E_A(b)c^*) = 0.$$

Hence $C \perp (A^\perp \cap B)$, and we get: A^\perp , $A^\perp \cap B$ and $A^\perp \cap C$ are orthogonal.

2 \Rightarrow 1: Assume 2. Let e_1, e_2 and e_3 be the projections on the orthogonal subspaces A^\perp , $A^\perp \cap B$ and $A^\perp \cap C$. Then

$$\begin{aligned} E_B E_C &= (e_1 + e_2)(e_1 + e_3) = e_1 = e_A \\ E_C E_B &= (e_1 + e_3)(e_1 + e_2) = e_1 = e_A \end{aligned}$$

2 \Leftrightarrow 3: Let $b \in B$ and $c \in C$, and decompose

$$b = b_1 + b_2, \quad c = c_1 + c_2, \quad b_1, c_1 \in A, \quad b_2 \in A^\perp \cap B, \quad c_2 \in A^\perp \cap C.$$

Then $E_A(b) = b_1$ and $E_A(c) = c_1$. Moreover

$$\langle b, c \rangle_{tr_D} = \langle b_1, c_1 \rangle_{tr_D} + \langle b_1, c_2 \rangle_{tr_D} + \langle b_2, c_1 \rangle_{tr_D} + \langle b_2, c_2 \rangle_{tr_D}. \quad (1.1)$$

Assume 2, then the last three terms of (1.1) vanish, and we get 3.

Assume 3, then for $b \in A^\perp \cap B$ and $c \in A^\perp \cap C$

$$\langle b, c \rangle_{tr_D} = \langle E_A(b), E_A(c) \rangle_{tr_D} = 0.$$

This proves 2. □

Lemma 1.2 *Let*

$$\begin{array}{ccc} C & \subset_L & D \\ \cup_K & & \cup_H \\ A & \subset_G & B \end{array} \quad (1.2)$$

be a commuting square of multi-matrix algebras with respect to a trace tr_D on D , and let $e \in A$ be an abelian projection with central support 1. Then

$$\begin{array}{ccc} eCe & \subset & eDe \\ \cup & & \cup \\ eAe & \subset & eBe \end{array} \quad (1.3)$$

form a commuting square with the same inclusion matrices as in (1.2) with respect to the trace $\frac{1}{tr_D(e)}tr_D$ on eDe . Moreover eAe is abelian.

Proof It is clear that (1.3) is a commuting square with respect to $\frac{1}{tr_D(e)}tr_D$, because E_A, E_B and E_C map eDe onto eAe, eBe resp. eCe . Since e has central support 1 in A , the least upper bound of $\{ueu^* \mid u \in A \text{ unitary}\}$ is 1. Hence e also has central support 1 in B, C and D . Therefore the map $z \mapsto ze$ is an isomorphism of the center $\mathcal{Z}(A)$ (resp. $\mathcal{Z}(B), \mathcal{Z}(C), \mathcal{Z}(D)$) onto the center $\mathcal{Z}(eAe)$ (resp. $\mathcal{Z}(eBe), \mathcal{Z}(eCe), \mathcal{Z}(eDe)$), and (up to these isomorphisms) the inclusion matrices of (1.3) are the same as those of (1.2), because any minimal projection in A (resp. B, C, D) is equivalent to a projection dominated by e , since e has central support 1 in all four algebras. □

Remark 1.3 Note that lemma 1.2 tells us, that a construction of a commuting square of multi-matrix algebra with given inclusion matrices, need only be concerned with an abelian algebra defining the smallest algebra, A , of the commuting square.

Lemma 1.4 *Let*

$$A \subset_G B \subset_H D \subset B(\mathcal{H})$$

be multi-matrix algebras with the commutant of D , D' , abelian, and let \mathcal{K} be a Hilbert space with $\dim(\mathcal{H}) = \dim(\mathcal{K})$. For $U \in B(\mathcal{H}, \mathcal{K})$ a unitary matrix, we put $A_1 = UAU^$, $B_1 = UBU^*$ and $D_1 = UDU^*$, then*

$$A_1 \subset_G B_1 \subset_H D_1 \subset B(\mathcal{H}),$$

and D'_1 is abelian.

Proof Trivial. □

Lemma 1.5 *Let \mathcal{H}, \mathcal{K} be finite dimensional Hilbert spaces. Let $A, D \subset B(\mathcal{H})$ and $A_1, D_1 \subset B(\mathcal{K})$ be four multi-matrix algebras, such that $A \subset D$ and $A_1 \subset D_1$, and such that the two inclusions have the same inclusion matrix, G , with respect to given isomorphisms $\Phi : \mathcal{Z}(A) \rightarrow \mathcal{Z}(A_1)$ and $\Psi : \mathcal{Z}(D) \rightarrow \mathcal{Z}(D_1)$. If furthermore A, A_1, D' and D'_1 are abelian, then there is a unitary $U \in B(\mathcal{H}, \mathcal{K})$, such that*

$$UAU^* = A_1 \quad \text{and} \quad UDU^* = D_1$$

and such that U implements the given isomorphisms $\Phi : \mathcal{Z}(A) \rightarrow \mathcal{Z}(A_1)$ and $\Psi : \mathcal{Z}(D) \rightarrow \mathcal{Z}(D_1)$.

Proof Since A and A_1 are abelian, Φ is an isomorphism of A onto A_1 . Since the inclusion matrices of $A \subset D$ and $A_1 \subset D_1$ are the same it follows from [Bra], that there is an isomorphism, Λ , of D onto D_1 , such that $\Lambda(A) = A_1$, $\Lambda|_A = \Phi$ and $\Lambda|_{\mathcal{Z}(D)} = \Psi$. But since D and D_1 are type I von Neumann algebras with abelian commutants, Λ is implemented by a unitary $U \in B(\mathcal{H}, \mathcal{K})$. (See [Dix], chap. III, § 3, sect. 2). □

Corollary 1.6 *If $A \subset_G B \subset_H D \subset B(\mathcal{H})$ and $A_1 \subset_K C \subset_L D_1 \subset B(\mathcal{K})$ are multi-matrix algebras, such that A, A_1, D' and D'_1 are abelian and $GH = KL$ with respect to given isomorphisms $\Phi : \mathcal{Z}(A) \rightarrow \mathcal{Z}(A_1)$ and $\Psi : \mathcal{Z}(D) \rightarrow \mathcal{Z}(D_1)$, then there exists a unitary $U \in B(\mathcal{H}, \mathcal{K})$, such that*

$$UA_1U^* = A, \quad UD_1U^* = D, \quad A \subset_K UCU^* \subset_L D \tag{1.4}$$

and

$$ad(U)|_{\mathcal{Z}(A)} = \Phi^{-1}, \quad ad(U)|_{\mathcal{Z}(D)} = \Psi^{-1}. \tag{1.5}$$

Proof Since $GH = KL$ and A, A_1, D' and D'_1 are abelian, lemma 1.5 produces a unitary $U \in B(\mathcal{H}, \mathcal{K})$, such that $UA_1U^* = A$, $UD_1U^* = D$ and such that (1.5) holds.

By lemma 1.4 we get the assertion

$$A \subset_K UCU^* \subset_L D.$$

□

We will now turn to the path model, which will allow us to build squares

$$\begin{array}{ccccc} C & & \subset_L & & D \\ \cup_K & & & & \cup_H \\ A & & \subset_G & & B \end{array}$$

of multi-matrix algebras with given inclusion matrices G , H , K and L .

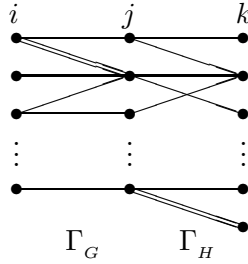
Let $G \in M_{nm}(\mathbb{Z})$, $H \in M_{mq}(\mathbb{Z})$, $K \in M_{np}(\mathbb{Z})$ and $L \in M_{pq}(\mathbb{Z})$ be matrices with non-negative entries, such that

$$GH = KL$$

and let Γ_G , Γ_H , Γ_K and Γ_L be the corresponding bi-partite graphs, i.e. the graphs with adjacency matrices

$$\begin{pmatrix} 0 & G \\ G^t & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & H \\ H^t & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & K \\ K^t & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & L \\ L^t & 0 \end{pmatrix}.$$

The Bratteli diagram for $A \subset B \subset D$ should be of the form



where the three columns have n , m and q vertices respectively. The paths from the left-hand column to the right-hand column are labeled by

$$\mathcal{S} = \{(i, j, k, \rho, \sigma) \mid G_{ij}H_{jk} \neq 0, 1 \leq \rho \leq G_{ij}, 1 \leq \sigma \leq H_{jk}\},$$

where ρ (resp. σ) labels the edges joining the same pair of vertices (i, j) (resp. (j, k)) in case of multiple edges.

Let \mathcal{H} be the Hilbert space of dimension $|\mathcal{S}|$ with orthonormal basis

$$\left\{ \xi_{ijk}^{(\rho, \sigma)} \mid (i, j, k, \rho, \sigma) \in \mathcal{S} \right\}.$$

For $x, y \in \mathcal{H}$ we let $x \otimes \bar{y}$ denote the rank one operator on \mathcal{H} given by

$$(x \otimes \bar{y})(z) = (z, y)x, \quad \text{for } z \in \mathcal{H}$$

Set

$$\begin{aligned} p_i &= \sum_{\substack{j, k, \rho, \sigma \\ (i, j, k, \rho, \sigma) \in \mathcal{S}}} \xi_{i, j, k}^{(\rho, \sigma)} \otimes \bar{\xi}_{i, j, k}^{(\rho, \sigma)}, \quad i = 1, \dots, n, \\ q_j &= \sum_{\substack{i, k, \rho, \sigma \\ (i, j, k, \rho, \sigma) \in \mathcal{S}}} \xi_{i, j, k}^{(\rho, \sigma)} \otimes \bar{\xi}_{i, j, k}^{(\rho, \sigma)}, \quad j = 1, \dots, m \end{aligned}$$

and

$$r_k = \sum_{\substack{i,j,\rho,\sigma \\ (i,j,k,\rho,\sigma) \in \mathcal{S}}} \xi_{i,j,k}^{(\rho,\sigma)} \otimes \bar{\xi}_{i,j,k}^{(\rho,\sigma)}, \quad k = 1, \dots, q.$$

Then the p'_i s (resp. the q'_j s and r'_k s) are orthogonal projections with sum 1. For fixed j the operators

$$f_{(i,\rho)(i',\rho')}^{(j)} = \sum_{\substack{k,\sigma \\ (i,j,k,\rho,\sigma) \in \mathcal{S} \\ (i',j,k,\rho',\sigma) \in \mathcal{S}}} \xi_{i,j,k}^{(\rho,\sigma)} \otimes \bar{\xi}_{i',j,k}^{(\rho',\sigma)}$$

form a set of matrix units for a full matrix algebra B_j with unit

$$\sum_{i,\rho} f_{(i,\rho)(i,\rho)}^{(j)} = q_j$$

and for fixed k the operators

$$g_{(i,j,\rho,\sigma)(i',j',\rho',\sigma')}^{(k)} = \xi_{i,j,k}^{(\rho,\sigma)} \otimes \bar{\xi}_{i',j',k}^{(\rho',\sigma')}$$

form a set of matrix units for a full matrix algebra D_k with unit

$$\sum_{i,j,\rho,\sigma} g_{(i,j,\rho,\sigma)(i,j,\rho,\sigma)}^{(k)} = r_k.$$

Set

$$\begin{aligned} A &= \bigoplus_i \mathbb{C}p_i, \\ B &= \bigoplus_j B_j, \\ D &= \bigoplus_k D_k, \end{aligned}$$

then one easily checks that $A \subset B \subset D$ with the inclusion matrices $A \subset_G B$ and $B \subset_H D$. Moreover A and the commutant, D' , of D are abelian algebras.

In the same way we can build algebras $A_1 \subset C_1 \subset D_1$ in $B(\mathcal{H}_1)$ with inclusion matrices $A_1 \subset_K B_1$ and $B_1 \subset_L D_1$, such that A_1 and D'_1 are abelian algebras. \mathcal{H}_1 is the Hilbert space with orthonormal basis

$$\left\{ \eta_{ilk}^{(\phi,\psi)} \mid (i,l,k,\phi,\psi) \in \mathcal{T} \right\}$$

where

$$\mathcal{T} = \{(i,l,k,\phi,\psi) \mid K_{il}L_{lk} \neq 0, 1 \leq \phi \leq K_{il}, 1 \leq \psi \leq L_{lk}\}.$$

Moreover $A_1 = \bigoplus_i \mathbb{C}p_i^1$, where

$$p_i^1 = \sum_{\substack{l,k,\phi,\psi \\ (i,l,k,\phi,\psi) \in \mathcal{T}}} \eta_{i,l,k}^{(\phi,\psi)} \otimes \bar{\eta}_{i,l,k}^{(\phi,\psi)}, \quad i = 1, \dots, n$$

and $C_1 = \bigoplus_l C_l^1$, $D_1 = \bigoplus_k D_k^1$, where the minimal central projections of C_1 and D_1 are given by

$$s_l^1 = \sum_{\substack{i,k,\phi,\psi \\ (i,l,k,\phi,\psi) \in \mathcal{T}}} \eta_{i,l,k}^{(\phi,\psi)} \otimes \bar{\eta}_{i,l,k}^{(\phi,\psi)}, \quad l = 1, \dots, p$$

$$r_k^1 = \sum_{\substack{i,l,\phi,\psi \\ (i,l,k,\phi,\psi) \in \mathcal{T}}} \eta_{i,l,k}^{(\phi,\psi)} \otimes \bar{\eta}_{i,l,k}^{(\phi,\psi)}, \quad k = 1, \dots, q$$

respectively. A set of matrix units for C_l^1 is given by

$$h_{(i,\phi)(i',\phi')}^{1(l)} = \sum_{\substack{k,\psi \\ (i,l,k,\phi,\psi) \in \mathcal{T} \\ (i',l,k,\phi',\psi) \in \mathcal{T}}} \eta_{i,l,k}^{(\phi,\psi)} \otimes \bar{\eta}_{i',l,k}^{(\phi',\psi)}.$$

By corollary 1.6 there is a unitary $U \in B(\mathcal{H}, \mathcal{H}_1)$ such that

$$U^* A_1 U = A, \quad U^* D_1 U = D \quad (1.6)$$

and

$$\begin{aligned} U^* p_i^1 U &= p_i, \quad i = 1, \dots, n \\ U^* r_k^1 U &= r_k, \quad k = 1, \dots, q. \end{aligned} \quad (1.7)$$

Moreover, by lemma 1.4 for any unitary $U \in B(\mathcal{H}, \mathcal{H}_1)$ satisfying (1.6) and (1.7),

$$\begin{array}{ccc} U^* C_1 U & \subset & D \\ \cup & & \cup \\ A & \subset & B \end{array} \quad (1.8)$$

is a square (not necessarily commuting) of multi-matrix algebras with the given inclusion matrices G, H, K and L . Furthermore A and D' are abelian.

Next we shall find a necessary and sufficient condition on U , for which (1.8) is a *commuting* square with respect to a given faithful trace tr_D on D .

Note first that (1.7) implies (1.6) because

$$\begin{aligned} A &= \text{span}\{p_i \mid i = 1, \dots, n\}, & A_1 &= \text{span}\{p_i^1 \mid i = 1, \dots, n\} \\ D' &= \text{span}\{r_k \mid k = 1, \dots, q\}, & D'_1 &= \text{span}\{r_k^1 \mid k = 1, \dots, q\} \end{aligned}$$

Assume that $U \in B(\mathcal{H}, \mathcal{K})$ is a unitary which satisfies (1.7). Since p_i is the projection on

$$\text{span} \left\{ \xi_{i,j,k}^{(\rho,\sigma)} \mid (i,j,k,\rho,\sigma) \in \mathcal{S}, \text{ (i fixed)} \right\}$$

and p_i^1 is the projection on

$$\text{span} \left\{ \eta_{i,l,k}^{(\phi,\psi)} \mid (i,l,k,\phi,\psi) \in \mathcal{T}, \text{ (i fixed)} \right\}$$

the condition $U^* p_i^1 U = p_i$ implies that

$$\left(\xi_{i,j,k}^{(\rho,\sigma)}, \eta_{i',l,k'}^{(\phi,\psi)} \right) = 0, \quad \text{when } i \neq i', \quad (1.9)$$

and similarly $U^* r_k^1 U = r_k$ implies that

$$\left(\xi_{i,j,k}^{(\rho,\sigma)}, \eta_{i',l,k'}^{(\phi,\psi)} \right) = 0, \quad \text{when } k \neq k'. \quad (1.10)$$

Hence the matrix, u , of U with respect to the ξ -basis of \mathcal{H} and the η -basis of \mathcal{K} , can be decomposed as a direct sum of unitary blocks

$$u = \bigoplus_{(i,k)} u^{(i,k)},$$

where the summation runs over all pairs (i, k) for which $(GH)_{i,k} = (KL)_{i,k} \neq 0$, and each block is given by

$$u^{(i,k)} = \left(u_{(j,\rho,\sigma)(l,\phi,\psi)}^{(i,k)} \right)_{\substack{(i,j,k,\rho,\sigma) \in \mathcal{S} \\ (i,l,k,\phi,\psi) \in \mathcal{T}}},$$

where

$$u_{(j,\rho,\sigma)(l,\phi,\psi)}^{(i,k)} = \left(U \xi_{i,j,k}^{(\rho,\sigma)}, \eta_{i,l,k}^{(\phi,\psi)} \right)$$

Note that each $u^{(i,k)}$ is a square matrix, with $(GH)_{i,k} = (KL)_{i,k}$ rows and columns.

Conversely, if $U \in B(\mathcal{H}, \mathcal{K})$ has a direct summand decomposition as described above, then (1.9) and (1.10) hold. Thus U maps $p_i(\mathcal{H})$ onto $p_i^1(\mathcal{K})$ and $q_k(\mathcal{H})$ onto $q_k^1(\mathcal{K})$, so (1.7) holds.

Assume in the following, that $U \in B(\mathcal{H}, \mathcal{K})$ satisfies (1.7). Let $\alpha_i, \beta_j, \gamma_l$ and δ_k be the trace-weights on $A, B, C = U^* C_1 U$ and D respectively.

For $d \in D$

$$E_A(d) = \sum_i \frac{\langle d, p_i \rangle_{tr_D}}{\langle p_i, p_i \rangle_{tr_D}} p_i$$

$\langle p_i, p_i \rangle_{tr_D} = tr_D(p_i) = \alpha_i$, since p_i is a minimal projection in A_i . Hence

$$E_A(d) = \sum_i \frac{1}{\alpha_i} tr_D(dp_i) p_i.$$

$tr_D(f_{(i,\rho)(i',\rho')}^{(j)} p_{i''}) = 0$ unless $i = i' = i''$ and $\rho = \rho'$, and since $f_{(i,\rho)(i,\rho)}^{(j)} \leq p_i$ we get

$$tr_D(f_{(i,\rho)(i,\rho)}^{(j)} p_i) = tr_D(f_{(i,\rho)(i,\rho)}^{(j)}) = \beta_j,$$

because $f_{(i,\rho)(i,\rho)}^{(j)}$ is a minimal projection in B_j . We now have

$$E_A(f_{(i,\rho)(i',\rho')}^{(j)}) = \begin{cases} \frac{\beta_j}{\alpha_i} p_i & \text{if } i = i' \text{ and } \rho = \rho' \\ 0 & \text{otherwise.} \end{cases} \quad (1.11)$$

Similarly

$$E_{A_1}(h_{(i,\phi)(i',\phi')}^{1(l)}) = \begin{cases} \frac{\gamma_l}{\alpha_i} p_i^1 & \text{if } i = i' \text{ and } \phi = \phi' \\ 0 & \text{otherwise,} \end{cases}$$

or equivalently

$$E_A(h_{(i,\phi)(i',\phi')}^{(l)}) = \begin{cases} \frac{\gamma_l}{\alpha_i} p_i & \text{if } i = i' \text{ and } \phi = \phi' \\ 0 & \text{otherwise,} \end{cases} \quad (1.12)$$

where we set

$$h_{(i,\phi)(i',\phi')}^{(l)} = U^* h_{(i,\phi)(i',\phi')}^{1(l)} U.$$

Note that $C = \bigoplus_l C_l$, where $C_l = U^* C_l^1 U$ and for fixed l , $h_{(i,\phi)(i',\phi')}^{(l)}$ form a set of matrix units for C_l .

By (1.11) and (1.12) we get

$$\mathrm{tr}_D(E_A(f_{(i,\rho)(i',\rho')}^{(j)})E_A(h_{(i'',\phi)(i''',\phi')}^{(l)})) = \begin{cases} \frac{\beta_j \gamma_l}{\alpha_i} & \text{if } i = i' = i'' = i''', \rho = \rho' \text{ and } \phi = \phi' \\ 0 & \text{otherwise.} \end{cases} \quad (1.13)$$

We will now compute

$$\mathrm{tr}_D(f_{(i,\rho)(i',\rho')}^{(j)} h_{(i'',\phi)(i''',\phi')}^{(l)}) = \sum_k \delta_k \mathrm{Tr}(f_{(i,\rho)(i',\rho')}^{(j)} h_{(i'',\phi)(i''',\phi')}^{(l)} r_k),$$

where Tr is the usual trace on each of the full matrix algebras D_k , $k = 1, \dots, q$.

Note that for $x, y, z, v \in \mathcal{H}$

$$\mathrm{Tr}((x \otimes \bar{y})(z \otimes \bar{v})) = (z, y) \mathrm{Tr}(x \otimes \bar{v}) = (z, y)(x, v). \quad (1.14)$$

We now get

$$f_{(i,\rho)(i',\rho')}^{(j)} r_k = \begin{cases} \sum_{\sigma=1}^{H_{jk}} \xi_{i,j,k}^{(\rho,\sigma)} \otimes \bar{\xi}_{i',j,k}^{(\rho',\sigma)} & \text{if } H_{jk} \neq 0 \\ 0 & \text{otherwise,} \end{cases}$$

$$r_k h_{(i,\phi)(i',\phi')}^{(l)} = \begin{cases} \sum_{\psi=1}^{L_{lk}} U^* \eta_{i,l,k}^{(\phi,\psi)} \otimes \overline{U^* \eta_{i',l,k}^{(\phi',\psi)}} & \text{if } L_{lk} \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$\begin{aligned} \mathrm{Tr}(f_{(i,\rho)(i',\rho')}^{(j)} h_{(i'',\phi)(i''',\phi')}^{(l)} r_k) &= \mathrm{Tr}(f_{(i,\rho)(i',\rho')}^{(j)} r_k r_k h_{(i'',\phi)(i''',\phi')}^{(l)}) = \\ &= \sum_{\sigma=1}^{H_{jk}} \sum_{\psi=1}^{L_{lk}} \mathrm{Tr}((\xi_{i,j,k}^{(\rho,\sigma)} \otimes \bar{\xi}_{i',j,k}^{(\rho',\sigma)})(U^* \eta_{i,l,k}^{(\phi,\psi)} \otimes \overline{U^* \eta_{i',l,k}^{(\phi',\psi)}})) = \\ &= \sum_{\sigma=1}^{H_{jk}} \sum_{\psi=1}^{L_{lk}} (U^* \eta_{i'',l,k}^{(\phi,\psi)}, \xi_{i',j,k}^{(\rho',\sigma)}) (\xi_{i,j,k}^{(\rho,\sigma)}, U^* \eta_{i''',l,k}^{(\phi',\psi)}) = \\ &= \sum_{\sigma=1}^{H_{jk}} \sum_{\psi=1}^{L_{lk}} (U^* \eta_{i',l,k}^{(\phi,\psi)}, \xi_{i',j,k}^{(\rho',\sigma)}) (\xi_{i,j,k}^{(\rho,\sigma)}, U^* \eta_{i,l,k}^{(\phi',\psi)}), \quad \text{if } i'' = i', i = i''' \text{ and } 0 \text{ otherwise.} \end{aligned}$$

Hence

$$\mathrm{tr}_D(f_{(i,\rho)(i',\rho')}^{(j)} h_{(i'',\phi)(i''',\phi')}^{(l)}) = \sum_k \sum_{\sigma=1}^{H_{jk}} \sum_{\psi=1}^{L_{lk}} \delta_k u_{(j,\rho,\sigma)(l,\phi',\psi)}^{(i,k)} \bar{u}_{(j,\rho',\sigma)(l,\phi,\psi)}^{(i',k)} \quad \text{if } i'' = i', i = i'''$$

and $\mathrm{tr}_D(f_{(i,\rho)(i',\rho')}^{(j)} h_{(i'',\phi)(i''',\phi')}^{(l)}) = 0$ otherwise.

Combining with (1.13) and lemma 1.1 3, we see that (1.8) is a commuting square if and only if

$$\sum_k \sum_{\sigma=1}^{H_{jk}} \sum_{\psi=1}^{L_{lk}} \delta_k u_{(j,\rho,\sigma)(l,\phi',\psi)}^{(i,k)} \overline{u}_{(j,\rho',\sigma)(l,\phi,\psi)}^{(i',k)} = \begin{cases} \frac{\beta_j \gamma_l}{\alpha_i} & \text{if } i = i', \rho = \rho', \phi = \phi' \\ 0 & \text{otherwise.} \end{cases} \quad (1.15)$$

Put

$$v_{(i,\rho,\phi)(k,\sigma,\psi)}^{(j,l)} = \sqrt{\frac{\alpha_i \delta_k}{\beta_j \gamma_l}} u_{(j,\rho,\sigma)(l,\phi,\psi)}^{(i,k)}. \quad (1.16)$$

Then

$$\sum_k \sum_{\sigma=1}^{H_{jk}} \sum_{\psi=1}^{L_{lk}} v_{(i,\rho,\phi)(k,\sigma,\psi)}^{(j,l)} \overline{v}_{(i',\rho',\phi')(k,\sigma,\psi)}^{(j,l)} = \delta_{(i,\rho,\phi)(i',\rho',\phi')},$$

where

$$\delta_{(i,\rho,\phi)(i',\rho',\phi')} = \begin{cases} 1 & \text{if } (i, \rho, \phi) = (i', \rho', \phi') \\ 0 & \text{otherwise.} \end{cases}$$

Let v be the matrix $v = \bigoplus_{(j,l)} v^{(j,l)}$, where

$$v^{(j,l)} = \left(v_{(i,\rho,\phi)(k,\sigma,\psi)}^{(j,l)} \right)_{\substack{(i,j,k,\rho,\sigma) \in \mathcal{S} \\ (i,l,k,\phi,\psi) \in \mathcal{T}}}$$

Then the commuting square condition (1.15) is equivalent to: Each summand, $v^{(j,l)}$, satisfies $v^{(j,l)} v^{(j,l)*} = 1$, which in turn means, that v is the matrix of an isometry. In particular $v^{(j,l)}$ has at least as many columns as rows.

Since

$$\begin{aligned} \# \text{ rows} &= \sum_i \sum_{\rho=1}^{G_{ij}} \sum_{\phi=1}^{K_{il}} 1 = \sum_i G_{ij} K_{il} = (G^t K)_{jl}, \\ \# \text{ columns} &= \sum_k \sum_{\sigma=1}^{H_{jk}} \sum_{\psi=1}^{L_{lk}} 1 = \sum_k H_{jk} L_{lk} = (H L^t)_{jl}, \end{aligned}$$

a necessary condition for (1.8) to be a commuting square is, that $G^t K \leq H L^t$ (element wise ordering).

All in all we have proved

Theorem 1.7 *With the notation introduced previously we have*

1. *The square (1.8) of multi-matrix algebras*

$$\begin{array}{ccc} U^* C_1 U & \subset_L & D \\ \cup_K & & \cup_H \\ A & \subset_G & B \end{array}$$

has the indicated inclusion matrices if and only if $U \in B(\mathcal{H}, \mathcal{K})$ is a unitary for which the matrix, u , with respect to the ξ -basis for \mathcal{H} and the η -basis for \mathcal{K} , is of the form

$$u = \bigoplus_{(i,k)} u^{(i,k)}$$

where

$$u^{(i,k)} = \left(u_{(j,\rho,\sigma)(l,\phi,\psi)}^{(i,k)} \right)_{\substack{(i,j,k,\rho,\sigma) \in \mathcal{S} \\ (i,l,k,\phi,\psi) \in \mathcal{T}}}.$$

2. The square (1.8) is a commuting square with respect to a given faithful trace, tr_D , on D if and only if the matrix $v = \bigoplus_{(j,l)} v^{(j,l)}$ with entries

$$v_{(i,\rho,\phi)(k,\sigma,\psi)}^{(j,l)} = \sqrt{\frac{\alpha_i \delta_k}{\beta_j \gamma_l}} u_{(j,\rho,\sigma)(l,\phi,\psi)}^{(i,k)}$$

is an isometry.

3. A necessary condition for (1.8) to be an commuting square is $G^t K \leq H L^t$ (element wise ordering).

Definition 1.8 If

$$\begin{array}{ccc} C & \subset_L & D \\ \cup_K & & \cup_H \\ A & \subset_G & B \end{array}$$

is a commuting square, with respect to the faithful trace, tr_D , on D , such that $G^t K = H L^t$ in addition to $GH = KL$, we say that the square is a symmetric commuting square

Remark 1.9 In the case (1.8) is a symmetric commuting square, the isometry v in theorem 1.7 2 becomes a unitary, so in the symmetric case we will refer to the condition in theorem 1.7 2 as the bi-unitary condition for the pair (u, v) .

Theorem 1.10 Let $G \in M_{nm}(\mathbb{Z})$, $H \in M_{mq}(\mathbb{Z})$, $K \in M_{np}(\mathbb{Z})$ and $L \in M_{pq}(\mathbb{Z})$ be matrices with non-negative entries, such that

$$GH = KL \quad \text{and} \quad G^t K = H L^t.$$

Then the following conditions are equivalent

- (a) There exists a (symmetric) commuting square

$$(A \subset B \subset D, \quad A \subset C \subset D, \quad \text{tr}_D)$$

of multi-matrix algebras, with inclusion matrices

$$\begin{array}{ccc} C & \subset_L & D \\ \cup_K & & \cup_H \\ A & \subset_G & B. \end{array}$$

(b) There exists a pair of matrices (u, v) satisfying the bi-unitary condition, i.e.

$$u = \bigoplus_{(i,k)} u^{(i,k)}, \quad v = \bigoplus_{(j,l)} v^{(j,l)}$$

where the direct summands

$$u^{(i,k)} = \left(u_{(j,\rho,\sigma)(l,\phi,\psi)}^{(i,k)} \right)_{\substack{(i,j,k,\rho,\sigma) \in \mathcal{S} \\ (i,l,k,\phi,\psi) \in \mathcal{T}}},$$

$$v^{(j,l)} = \left(v_{(i,\rho,\phi)(k,\sigma,\psi)}^{(j,l)} \right)_{\substack{(i,j,k,\rho,\sigma) \in \mathcal{S} \\ (i,l,k,\phi,\psi) \in \mathcal{T}}}$$

are unitary matrices and

$$v_{(i,\rho,\phi)(k,\sigma,\psi)}^{(j,l)} = \sqrt{\frac{\alpha_i \delta_k}{\beta_j \gamma_l}} u_{(j,\rho,\sigma)(l,\phi,\psi)}^{(i,k)} \quad (1.17)$$

Here $\alpha_i, \beta_j, \gamma_l$ and δ_k are the trace weights on A, B, C resp. D coming from tr_D , and the indices $i, j, k, l, \rho, \sigma, \phi$ and ψ are as in theorem 1.7.

Proof

(b) \Rightarrow (a) follows from theorem 1.7 and remark 1.9.

(a) \Rightarrow (b). Assume (a). Then by reducing with an abelian projection e in A , with central support 1, as in lemma 1.2, we get a new commuting square,

$$\begin{array}{ccc} \mathcal{C} & \subset_L & \mathcal{D} \\ \cup_K & & \cup_H \\ \mathcal{A} & \subset_G & \mathcal{B} \end{array} \quad (1.18)$$

of multi-matrix algebras, with the same inclusion matrices, such that \mathcal{A} is abelian. Note that the reduction with e does not change the factor $\sqrt{\alpha_i \delta_k / \beta_j \gamma_l}$ in (1.17), because $\alpha_i, \beta_j, \gamma_l$ and δ_k are all multiplied with the same constant $(\text{tr}_{\mathcal{D}}(e))^{-1}$.

Next we can represent \mathcal{D} on a Hilbert space, such that the commutant, \mathcal{D}' , is abelian.

As in the proof of lemma 1.5, the inclusion $\mathcal{A} \subset_G \mathcal{B} \subset_H \mathcal{D}$ is spatially isomorphic to any other inclusion of multi-matrix algebras, $A \subset_G B \subset_H D$, with the same inclusion matrices, for which A and D' are abelian. In particular it is spatially isomorphic to $A \subset_G B \subset_H D$ coming from the path construction described previously. Similarly $\mathcal{A} \subset_K \mathcal{C} \subset_L \mathcal{D}$ is spatially isomorphic to $A_1 \subset_K C_1 \subset_L D_1$ coming from the path construction. Hence (1.18) is spatially isomorphic to (1.8) for some unitary $u \in B(\mathcal{H}, \mathcal{K})$. Therefore (a) \Rightarrow (b) follows from theorem 1.7 and remark 1.9. \square

Proposition 1.11 If

$$(A \subset_G B \subset_H D, \quad A \subset_K C \subset_L D, \quad \text{tr}_D)$$

is a symmetric commuting square, such that the Bratteli diagrams $\Gamma_G, \Gamma_H, \Gamma_K$ and Γ_L are connected, then

- (I) $\|K\| = \|H\|$. Moreover tr_D is the Markov trace of the embedding $C \subset D$, and $\text{tr}_D|_B$ is the Markov trace of the embedding $A \subset B$.
- (II) $\|G\| = \|L\|$. Moreover tr_D is the Markov trace of the embedding $B \subset D$, and $\text{tr}_D|_C$ is the Markov trace of the embedding $A \subset C$.

Proof By the assumptions $GH = KL$ and $G^tK = HL^t$. Let $u = \bigoplus u^{(i,k)}$ and $v = \bigoplus v^{(j,l)}$ be as in theorem 1.10. Let

$$N(i, j, k, l, \rho, \sigma, \phi, \psi) = \alpha_i \delta_k \left| u_{(j, \rho, \sigma)(l, \phi, \psi)}^{(i, k)} \right|^2 = \beta_j \gamma_l \left| v_{(i, \rho, \phi)(k, \sigma, \psi)}^{(j, l)} \right|^2$$

if $(i, j, k, \rho, \sigma) \in \mathcal{S}$ and $(i, l, k, \phi, \psi) \in \mathcal{T}$, and let $N(i, j, k, l, \rho, \sigma, \phi, \psi) = 0$ otherwise. Since u and v are unitary we get

$$\sum_j \sum_{\rho=1}^{G_{ij}} \sum_{\sigma=1}^{H_{jk}} N(i, j, k, l, \rho, \sigma, \phi, \psi) = \begin{cases} \alpha_i \delta_k & \text{if there exists a path } k - l - i \\ 0 & \text{otherwise} \end{cases} \quad (1.19)$$

$$\sum_l \sum_{\phi=1}^{K_{il}} \sum_{\psi=1}^{L_{lk}} N(i, j, k, l, \rho, \sigma, \phi, \psi) = \begin{cases} \alpha_i \delta_k & \text{if there exists a path } i - j - k \\ 0 & \text{otherwise} \end{cases} \quad (1.20)$$

$$\sum_i \sum_{\rho=1}^{G_{ij}} \sum_{\phi=1}^{K_{il}} N(i, j, k, l, \rho, \sigma, \phi, \psi) = \begin{cases} \beta_j \gamma_l & \text{if there exists a path } j - k - l \\ 0 & \text{otherwise} \end{cases} \quad (1.21)$$

$$\sum_k \sum_{\sigma=1}^{H_{jk}} \sum_{\psi=1}^{L_{lk}} N(i, j, k, l, \rho, \sigma, \phi, \psi) = \begin{cases} \beta_j \gamma_l & \text{if there exists a path } l - i - j \\ 0 & \text{otherwise} \end{cases} \quad (1.22)$$

where the term “path” refers to paths on the graphs Γ_X , $X = G, H, K, L$, so f. inst. there exists a path $k - i - l$ if and only if $L_{lk} \neq 0$ and $K_{il} \neq 0$.

Assume that there is an edge $l - k$, i.e. $L_{lk} \neq 0$. By (1.19) we get

$$\sum_{\substack{i \\ K_{il} \neq 0}} \sum_{\phi=1}^{K_{il}} \sum_j \sum_{\rho=1}^{G_{ij}} \sum_{\sigma=1}^{H_{jk}} N(i, j, k, l, \rho, \sigma, \phi, \psi) = \sum_{\substack{i \\ K_{il} \neq 0}} \sum_{\phi=1}^{K_{il}} \alpha_i \delta_k = \sum_i \alpha_i K_{il} \delta_k = (K^t \alpha)_l \delta_k.$$

Hence

$$\sum_j \sum_i \sum_{\sigma=1}^{H_{jk}} \sum_{\rho=1}^{G_{ij}} \sum_{\phi=1}^{K_{il}} N(i, j, k, l, \rho, \sigma, \phi, \psi) = \begin{cases} (K^t \alpha)_l \delta_k & \text{if } L_{lk} \neq 0 \\ 0 & \text{otherwise} \end{cases} \quad (1.23)$$

Summing over j and σ in (1.21) gives

$$(1.23) = \begin{cases} (H^t \beta)_k \gamma_l & \text{if } L_{lk} \neq 0 \\ 0 & \text{otherwise,} \end{cases}$$

and we have

$$\frac{(H^t\beta)_k}{\delta_k} = \frac{(K^t\alpha)_l}{\gamma_l} \quad \text{if } L_{lk} \neq 0.$$

Since Γ_L is connected, we then have

$$\frac{(H^t\beta)_k}{\delta_k} = \frac{(K^t\alpha)_l}{\gamma_l} \quad \text{for all } l, k,$$

and hence we can find $\mu > 0$ such that

$$H^t\beta = \mu\delta \quad \text{and} \quad K^t\alpha = \mu\gamma \quad (1.24)$$

and since

$$\beta = H\delta \quad \text{and} \quad \alpha = K\gamma$$

we have

$$H^tH\delta = \mu\delta \quad \text{and} \quad K^tK\gamma = \mu\gamma, \quad (1.25)$$

which shows that δ is the Perron–Frobenius eigenvector for H^tH , and that γ is the Perron–Frobenius eigenvector for K^tK , both corresponding to the same eigenvalue. Hence $\|H\| = \|K\|$. Using [G.H.J.] theorem 2.1.3(i), (1.24) and (1.25) imply the Markov trace assertions of (I).

The proof of (II) follows the same lines, considering (1.20) and (1.22) for fixed i, j such that $G_{ij} \neq 0$. \square

Let

$$\begin{array}{ccc} B_0 & \subset_L & B_1 \\ \cup_K & & \cup_H \\ A_0 & \subset_G & A_1 \end{array}$$

be a symmetric commuting square. By the Markov trace properties of proposition 1.11 and [G.H.J.] lemma 4.2.4 and proposition 2.4.1, we can use the fundamental construction to obtain a ladder of multi–matrix algebras

$$\begin{array}{ccccccc} B_0 & \subset_L & B_1 & \subset_{L^t} & B_2 & \subset_L & B_3 & \cdots \\ \cup_K & & \cup_H & & \cup_K & & \cup_H & \\ A_0 & \subset_G & A_1 & \subset_{G^t} & A_2 & \subset_G & A_3 & \cdots \end{array}$$

By [G.H.J.] corollary 4.2.3 each

$$\begin{array}{ccc} B_i & \subset & B_{i+1} \\ \cup & & \cup \\ A_i & \subset & A_{i+1} \end{array}$$

is a commuting square.

Theorem 1.5 and theorem 1.6 of [Wen2] and corollary 17.5 now give

Proposition 1.12 *If*

$$\begin{array}{ccc} B_0 & \subset_L & B_1 \\ \cup_K & & \cup_H \\ A_0 & \subset_G & A_1 \end{array}$$

is a symmetric commuting square, with $\Gamma_G, \Gamma_H, \Gamma_K$ and Γ_L connected, and if Γ_H or Γ_K has a vertex which is connected to only one other vertex of valency 1, then there exists an irreducible subfactor, A , of the hyperfinite II_1 -factor, R , such that

$$[R : A] = \|H\|^2 = \|K\|^2$$

2 Special Symmetric Commuting Squares

In this section we shall look at which bi-partite connected graphs Γ with $\|\Gamma\|^2 \in (4, 5)$ can define a commuting square of the form

$$\begin{array}{ccc} C & \subset_{nG^t} & D \\ \cup_G & & \cup_{G^t} \\ A & \subset_{nG} & B \end{array} \quad (2.1)$$

where G is the Bratteli diagram of a bi-partition of Γ , and we shall compute the bi-unitarity condition for these graphs.

Note that if the above diagram is a commuting square, then it necessarily is a symmetric commuting square. Hence by proposition 1.11 we know that the only tracial weights, which will satisfy the Markov-trace conditions on the inclusions $A \subseteq B$, $B \subseteq D$, $A \subseteq C$ and $C \subseteq D$ are those determined by the Perron–Frobenius eigenvector of G . We shall now inductively define an n 'th degree polynomial $R_n(t)$, which will be of great help in the discussion to come.

Definition 2.1 *Let $t \in \mathbb{R}$ and put $R_0(t) = 1$, $R_1(t) = t$ and define inductively*

$$R_n(t) = tR_{n-1}(t) - R_{n-2}(t), \quad n \geq 2.$$

Note that

$$R_n(t) = \begin{cases} \frac{\sin((n+1)x)}{\sin(x)} & \text{if } t = 2 \cos(x), \quad x \in (0, \frac{\pi}{2}) \\ n + 1 & \text{if } t = 2 \\ \frac{\sinh((n+1)x)}{\sinh(x)} & \text{if } t = 2 \cosh(x), \quad x > 0 \end{cases}$$

Remark 2.2 If we want to determine whether a symmetric commuting square of the form

$$\begin{array}{ccc} C & \subset_L & D \\ \cup_K & & \cup_H \\ A & \subset_G & B \end{array} \quad (2.2)$$

exists, the bi-unitary condition tells us that we have to show the existence of the two matrices u and v of (1.9). The way we will usually proceed to prove or disprove the existence of u and/or v is as follows.

- (a) Determine all the cycles of length four in the diagram 2.2. These cycles label the entries of u and v , if such matrices exist, and so we can group these to obtain the labeling of the blocks which u and v must consist of.
- (b) Using:
 1. The block structure determined above,
 2. The transition rule that u and v must satisfy,

3. The matrix consisting of the moduli squared of the entries of a unitary matrix has sum of a row or a column equal to 1,

we are, in most cases, able to determine the the values which the moduli of the entries of u and v must have, if a solution exists.

- (c) Determine whether “phases” on each entry of the, in b. determined, matrices of moduli can be found, to make these into the unitary matrices u and v .

Lemma 2.3 *Let Γ be a bi-partite connected graph without multiple edges and without any cycles of length 4. Let G be the adjacency matrix of a bi-partition of Γ .*

Consider the diagram

$$\begin{array}{ccc} C & \subset_{nG^t} & D \\ \cup_G & & \cup_{G^t} \\ A & \subset_{nG} & B \end{array}$$

with Bratteli diagrams determined by the bi-partition of Γ . The labels of the minimal central projections will be

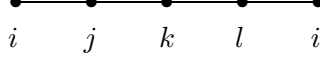
$$\begin{array}{ccc} \{l\} & \equiv & \{k\} \\ | & & | \\ \{i\} & \equiv & \{j\} \end{array}$$

Denote the unitary part of the bi-unitary condition, corresponding to i and k fixed, by u and the part corresponding to j and l fixed by v . If there is a solution to (2.1) for n we have, that for i and k fixed (resp. j and l fixed) the block of u (resp. v) labeled by paths of the form $i - j - k - l$, j, l varying (resp. i, k varying), is proportional to an $n \times n$ unitary, and the proportionality constant is the modulus of the corresponding entry of the matrix discussed in remark 2.2 (a) and (b), in the case $n = 1$.

Proof Let i, j, k and l be such that $G_{ij}G_{kj}G_{kl}G_{il} \neq 0$, i.e. there exists a path $i - j - k - l - i$ in the diagram

$$\begin{array}{ccc} \{l\} & \equiv & \{k\} \\ | & & | \\ \{i\} & \equiv & \{j\} \end{array}$$

If i and k do not label the same vertex of Γ , then j and l must label the same vertex of Γ , since otherwise



would define a cycle of length 4 on Γ .

In particular the only paths fixing i and k are of the form $(ij)_\nu(kl)_\mu$, where $(ij)_\nu$ resp. $(kl)_\mu$ denotes one of the n edges joining i to j , resp. k to l , in the diagram. The part of u , $u^{(i,k)}$, corresponding to i and k fixed is then an $n \times n$ unitary, and the corresponding part of v is

$$\sqrt{\frac{\alpha_i \delta_k}{\beta_j \gamma_l}} \left(u^{(i,k)} \right)^t$$

and hence proportional to a unitary. The proportionality constant is the modulus of the corresponding entry in the case $n = 1$.

If j and l do not label the same vertex of Γ a similar argument is valid.

If i and k denote the same vertex of Γ , and j and l also denote the same vertex of Γ , there are two essentially different situations:

- a) i is only connected to j and vice versa. This implies that $\Gamma = \bullet \text{---} \bullet$, and hence $\|\Gamma\| = 2$.
- b) Either i or j is connected to some other vertex.

Assume that there exists $j_1 \neq j$ s.t. $G_{ij_1} \neq 0$, and let $\mathcal{J} = \{j | G_{ij} \neq 0\}$.

The only paths joining $i - j - k - l - i$, which fix i and k , are of the form

$$i - j_1 - k - j_2 - i, \text{ where } j_1, j_2 \in \mathcal{J}.$$

If $j_1 \neq j_2$, we know that the part of v , corresponding to j_1 and j_2 fixed, is an $n \times n$ unitary $v^{(j_1, j_2)}$, and hence the corresponding part of u is proportional to a unitary. Hence the part of u corresponding to i and k fixed, is of the form

$$u^{(i,k)} = \begin{pmatrix} X_{j_1 j_1} & X_{j_1 j_2} & \cdots & X_{j_1 j_m} \\ X_{j_2 j_1} & \ddots & & \\ \vdots & & & \\ X_{j_m j_1} & & & X_{j_m j_m} \end{pmatrix}$$

with all the off-diagonal blocks proportional to $n \times n$ unitaries. This implies that the diagonal blocks are also proportional to $n \times n$ unitaries, i.e.

$$X_{j_q j_q} = k_{j_q j_q} C_{j_q j_q}, \quad C_{j_q j_q} \in U(M_n(\mathbb{C})).$$

Since $u^{(i,k)}$ is unitary, the matrix formed by all moduli squared of the entries of $u^{(i,k)}$ is doubly stochastic. Summing over a row in this matrix yields

$$k_{j_q j_q}^2 + \sum_{j \neq j_q} \frac{\beta_j \beta_{j_q}}{\alpha_i^2} = 1,$$

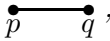
which is also the equation that occurs in the case $n = 1$. Hence k_{jqjq} equals the modulus of the corresponding entry in the case $n = 1$. \square

Corollary 2.4 *Let Γ be a connected bi-partite graph without multiple edges, and consider a fixed path $i - j - k - l - i$ in the diagram (2.1). If one of the edges in Γ labeled by $i - j$ or $j - k$ and one of the edges in Γ labeled by $i - l$ or $l - k$ are not edges in a cycle of length 4, then the statement of lemma 2.3 holds for the path $i - j - k - l - i$.*

Proof Copy the proof of lemma 2.3. \square

For the discussion to follow we let Γ be a bi-partite connected graph with $\|\Gamma\|^2 \in (4, 5)$, such that there exists a commuting square of the form (2.1) for some n .

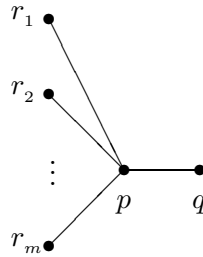
Remark 2.5 The graph  has Perron–Frobenius eigenvalue $\sqrt{5}$, so Γ does not contain multiple edges.

Lemma 2.6 *Assume Γ has an edge which is not part of a 4-cycle. Denote this edge by , and the corresponding Perron–Frobenius coordinates by α_p and α_q . Then either*

$$\frac{\alpha_p}{\alpha_q} \geq e^x \text{ or } \frac{\alpha_p}{\alpha_q} \leq e^{-x},$$

where $\|\Gamma\| = e^x + e^{-x}$.

Proof If there is a solution for $n = 1$ we have: At p the graph will look like



Let α_{r_i} denote the Perron–Frobenius coordinate corresponding to r_i , $i = 1, \dots, m$. Let u and v denote the two unitaries of the bi-unitarity condition as in remark 1.9. In the labels of cycles in $A \xrightarrow{G} B \xrightarrow{G^t} D \xrightarrow{G} C \xrightarrow{G^t} A$ we may substitute B labels for C labels. Consider the cycles of the form $p?p?$. These are

$$pqppq, pqpr_1, \dots, pqpr_m, pr_1pq, pr_1pr_2, \dots, pr_mpr_m.$$

Since the only path of length 2 from q to r_i passes through p , the modulus of the corresponding entry in say u (or v depending on which bi-partition is chosen) is 1 for all paths $pr_i pq$. Hence the

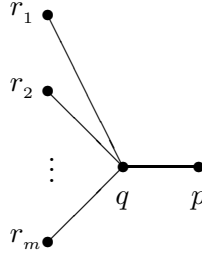
modulus squared of the corresponding entry of v is $\frac{\alpha_q \alpha_{r_i}}{\alpha_p^2}$. By double stochastic property of the matrix formed by the moduli squared of the entries of v , we get, that the modulus squared, of the entry corresponding to pqq is $1 - \frac{\alpha_q}{\alpha_p^2} \sum_i \alpha_{r_i}$. Since $\sum \alpha_{r_i} + \alpha_q = \lambda \alpha_p$, we have

$$0 \leq 1 - \frac{\alpha_q}{\alpha_p^2} \sum_i \alpha_{r_i} \leq 1 \Leftrightarrow \quad (2.3)$$

$$\left(\frac{\alpha_q}{\alpha_p}\right)^2 - \lambda \left(\frac{\alpha_q}{\alpha_p}\right) + 1 \in [0, 1] \Leftrightarrow \left(\frac{\alpha_q}{\alpha_p}\right) \in \left[\frac{1}{\lambda}, \frac{\lambda - \sqrt{\lambda^2 - 4}}{2}\right] \cup \left[\frac{\lambda + \sqrt{\lambda^2 - 4}}{2}, \lambda\right] = \left[\frac{1}{\lambda}, e^{-x}\right] \cup [e^x, \lambda].$$

For general n , the statement of corollary 2.4 will hold for the path pqq , and we have that the corresponding entries of u and v are proportional to $n \times n$ unitaries, with proportionality constants equal the entries of the above determined matrices. When we sum the moduli-squared of a row or column of an $n \times n$ unitary we get 1, and hence also obtain the inequality (2.3) in this case. \square

Assume Γ has no 4-cycles, and look at an edge $\overset{p}{\bullet} \text{---} \overset{q}{\bullet}$. We may assume $\frac{\alpha_p}{\alpha_q} \geq e^x$ (by lemma 2.6). If the number of vertices adjacent to q is $m + 1$, we have

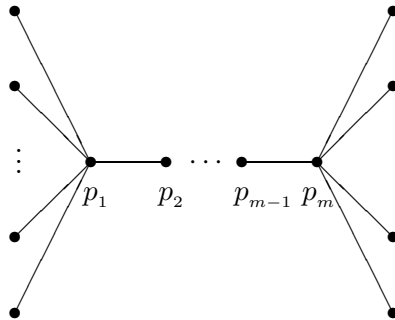


Since $\lambda \alpha_{r_i} \geq \alpha_q$, $i = 1, \dots, m$ and $\alpha_p + \sum_i \alpha_{r_i} = \lambda \alpha_q$ we have

$$\frac{m}{\lambda} + e^x \leq \lambda \Leftrightarrow m \leq \lambda e^{-x} = 1 + e^{-2x} \leq 2.$$

Hence the valency of q is either 1 or 2.

If Γ has at least two vertices of valency at least 3, Γ looks like



Since the valency of p_1 is three, we must have $\frac{\alpha_{p_2}}{\alpha_{p_1}} \leq e^{-x}$, by the above argument. Since the α'_p s are the Perron–Frobenius coordinates, we have $\lambda \alpha_{p_k} = \alpha_{p_{k-1}} + \alpha_{p_{k+1}}$, $2 \leq k \leq m - 1$.

We have

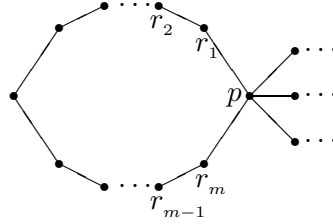
$$\frac{\alpha_{p_3}}{\alpha_{p_2}} = \frac{\lambda\alpha_{p_2} - \alpha_{p_1}}{\alpha_{p_2}} = \lambda - \frac{\alpha_{p_1}}{\alpha_{p_2}} = e^x + e^{-x} - \frac{\alpha_{p_1}}{\alpha_{p_2}} \leq e^{-x},$$

so $\frac{\alpha_{p_3}}{\alpha_{p_2}} \leq e^{-x}$. Assume now that $\frac{\alpha_{p_{k+1}}}{\alpha_{p_k}} \leq e^{-x}$, for all $k \leq j \leq m-2$. Then

$$\frac{\alpha_{p_{j+1}}}{\alpha_{p_j}} = \frac{\lambda\alpha_{p_j} - \alpha_{p_{j-1}}}{\alpha_{p_j}} = \lambda - \frac{\alpha_{p_{j-1}}}{\alpha_{p_j}} = e^x + e^{-x} - \frac{\alpha_{p_{j-1}}}{\alpha_{p_j}} \leq e^{-x},$$

and hence by induction: $\frac{\alpha_{p_{j+1}}}{\alpha_{p_j}} \leq e^{-x}$, for all $1 \leq j \leq m-1$. In particular we have $\frac{\alpha_{p_m}}{\alpha_{p_{m-1}}} \leq e^{-x}$, which contradicts that the valency of p_m is three. Hence we may conclude, that if Γ has no cycles of length 4 then Γ has at most one vertex of valency ≥ 3 .

If Γ has no cycles of length 4, but longer cycles, the discussion so far gives, that Γ is of the form



We must have

$$\frac{\alpha_{r_1}}{\alpha_p} \leq e^{-x} \quad \text{and} \quad \frac{\alpha_{r_m}}{\alpha_p} \leq e^{-x},$$

and, if we use the previous argument starting in r_1 , we obtain

$$\frac{\alpha_{r_{k+1}}}{\alpha_{r_k}} \leq e^{-x}, \quad k = 1, \dots, m-1.$$

If we start in r_m we also get

$$\frac{\alpha_{r_k}}{\alpha_{r_{k+1}}} \leq e^{-x}, \quad k = 1, \dots, m-1.$$

Hence

$$e^x \leq \frac{\alpha_{r_{k+1}}}{\alpha_{r_k}} \leq e^{-x}, \quad k = 1, \dots, m-1,$$

and since $x \geq 0$ we must have $x = 0$, and hence $\|\Gamma\| = 2$.

Definition 2.7 We say that Γ is a m -star, if Γ is connected, and has a “central” vertex p , of valency m , and m rays of the form $p \text{---} \bullet \text{---} \dots \text{---} \bullet$ with k_i vertices (not counting p), $i = 1, \dots, m$. We will denote a m -star by $S(k_1, k_2, \dots, k_m)$, $k_1 \leq k_2 \leq \dots \leq k_m$.

Assume that Γ has at least one cycle of length 4.

Since

1.

$$\left\| \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \right\|^2 = 3 + \sqrt{5} > 5$$

each vertex in a 4-cycle has valency at most 3.

2.

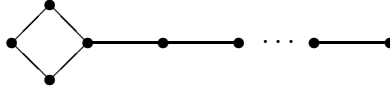
$$\left\| \begin{array}{c} \bullet \quad \bullet \\ \parallel \quad \parallel \\ \bullet \quad \bullet \end{array} \right\|^2 = \frac{\sin^2 \frac{3\pi}{7}}{\sin^2 \frac{\pi}{7}} > 5$$

3.

$$\left\| \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \right\|^2 = 5$$

Hence at most one vertex in a 4-cycle has valency 3.

By 1, 2 and 3 an “extra” edge on a vertex in a 4-cycle cannot be edge in a 4-cycle. The argument used to show that there is at most one vertex of valency 3, in the case with no 4-cycles, only depends on the fact that we have an edge which not part of a 4-cycle. Hence Γ must be of the form



$k + 4$ vertices, and $k \geq 1$ since

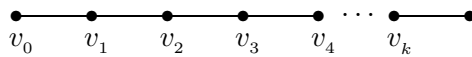
$$\left\| \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \right\|^2 = 4.$$

We shall call the above graph $\text{kite}(k)$.

Note that the Perron–Frobenius eigenvalue of an m –star with all rays of length 1 is \sqrt{m} , so our discussion has excluded all graphs except 3-stars, 4-stars and $\text{kite}(k)$, $k \in \mathbb{N}$.

Since it turns out that the computation of the bi-unitarity condition for 3-stars and 4-stars, defining the inclusions in (2.1), is almost the same, we will use the rest of this section to compute the condition for an m –star, $m \geq 2$.

Lemma 2.8 *If Γ is a graph, with part of Γ a ray (in the sense of the definition of m –stars)*



then the coordinate of the Perron–Frobenius vector corresponding to v_i is proportional to $R_i(\lambda)$, $i = 0, \dots, k$, where λ denotes the Perron–Frobenius eigenvalue of Γ .

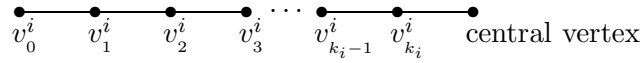
Proof Let α_0 denote the Perron–Frobenius coordinate at v_0 . Then $\lambda\alpha_0 = \alpha_1$, and we have $\alpha_1 = R_1(\lambda)\alpha_0$.

Assume $\alpha_j = R_j(\lambda)\alpha_0$, for $j \leq n < k$. We then have

$$\alpha_{n+1} = \lambda\alpha_n - \alpha_{n-1} = (\lambda R_n(\lambda) - R_{n-1}(\lambda))\alpha_0 = R_{n+1}(\lambda)\alpha_0.$$

□

If we apply lemma 2.8 to each ray of $S(k_1, k_2, \dots, k_m)$ we have, for the i 'th ray,



and the Perron–Frobenius coordinate at v_j^i can be chosen to

$$\frac{R_j(\lambda)}{R_{k_i}(\lambda)}, \quad j = 1, \dots, k_i, \quad i = 1, \dots, m, \quad (2.4)$$

and the central vertex is assigned the value 1. If the defined coordinates shall define an eigenvector on all of Γ , they must match up at the central vertex to give

$$\sum_{i=1}^m \frac{R_{k_i-1}(\lambda)}{R_{k_i}(\lambda)} = \lambda,$$

that is, the Perron–Frobenius eigenvalue of $S(k_1, k_2, \dots, k_m)$ is the largest solution to

$$\sum_{i=1}^m \frac{R_{k_i-1}(t)}{R_{k_i}(t)} = t. \quad (2.5)$$

The above equation will be referred to as the *eigenvalue equation*.

To determine the bi-unitarity condition for the diagram (2.1) with G the adjacency matrix of an m -star, lemma 2.3 suggests that we first look at the case $n = 1$, and determine the moduli of the matrix entries. For $n = 1$ cycles in the diagram corresponds to cycles on Γ of length 4. In this case a cycle in the diagram (2.1) is completely determined by listing the edges $\overset{\bullet}{i} \text{---} \overset{\bullet}{\ell}$, $\overset{\bullet}{j} \text{---} \overset{\bullet}{k}$ of Γ . Hence we can picture the blocks in a diagram with the edges $\overset{\bullet}{i} \text{---} \overset{\bullet}{\ell}$ on the vertical axis, and the edges $\overset{\bullet}{j} \text{---} \overset{\bullet}{k}$ on the horizontal axis.

We will first look at what happens on a ray of the m -star, so far from the central vertex, that the cycles do not involve the central vertex. To indicate that a vertex is in the “ i ” or “ k ” corner of the diagram (2.1), we will draw it as a \star . The vertices of the “ j ” and “ ℓ ” corners of the diagram are drawn as a \bullet .

We have chosen to put the central vertex in the “ i ” and “ k ” corners of the diagram. Had we chosen to put the central vertex in the other two corners, the only difference in the discussion to come

would be, that all the u matrices would be v matrices and vice versa. Since the graphs do not contain multiple edges we do not need the σ, ρ, ϕ and ψ labels in the bi-unitary condition.

The block structure of u and v is pictured in the diagrams by the thick lines.

The moduli in the boxes are determined using the following facts

1. If we have a 1×1 block, the modulus of this element is 1, since it has to be a 1×1 unitary.
2. In a 2×2 block the moduli have to be of the following form

$$\begin{pmatrix} a & \sqrt{1-a^2} \\ \sqrt{1-a^2} & a \end{pmatrix}, \quad 0 \leq a \leq 1.$$

3. The transition formula between u and v

$$\begin{aligned} |v_{ik}^{jl}| &= \sqrt{\frac{\alpha_i \delta_k}{\beta_j \gamma_l}} |u_{jl}^{ik}| \\ |u_{jl}^{ik}| &= \sqrt{\frac{\beta_j \gamma_l}{\alpha_i \delta_k}} |v_{ik}^{jl}| \end{aligned}$$

1, 2, and 3 allows us to determine the modulus of a entry in each 2×2 block, and hence by 2 of every entry in the block.

Let v_c denote the central vertex of $S(k_1, k_2, \dots, k_m)$ and consider the j' th ray

$$\begin{array}{ccccccccccc} \star & \bullet & \star & \bullet & \cdots & \star & \bullet & \star & & & \\ v_0^j & v_1^j & v_2^j & v_3^j & & v_{k_j-1}^j & v_{k_j}^j & c & & & k_j \text{ even} \end{array}$$

$$\begin{array}{ccccccccccc} \bullet & \star & \bullet & \star & \cdots & \star & \bullet & \star & & & \\ v_0^j & v_1^j & v_2^j & v_3^j & & v_{k_j-1}^j & v_{k_j}^j & c & & & k_j \text{ odd} \end{array}$$

Ray of even length. The blocks of u .

	v_6^j	v_5^j	v_4^j	v_3^j	v_2^j	v_1^j	v_0^j
v_6^j	$\frac{R_0(\lambda)}{R_6(\lambda)}$	1					
v_5^j	1	$\frac{R_0(\lambda)}{R_4(\lambda)}$	$\frac{\sqrt{R_3(\lambda)R_5(\lambda)}}{R_4(\lambda)}$				
v_4^j		$\frac{\sqrt{R_3(\lambda)R_5(\lambda)}}{R_4(\lambda)}$	$\frac{R_0(\lambda)}{R_4(\lambda)}$	1			
v_3^j			1	$\frac{R_0(\lambda)}{R_2(\lambda)}$	$\frac{\sqrt{R_1(\lambda)R_3(\lambda)}}{R_2(\lambda)}$		
v_2^j				$\frac{\sqrt{R_1(\lambda)R_3(\lambda)}}{R_2(\lambda)}$	$\frac{R_0(\lambda)}{R_2(\lambda)}$	1	
v_1^j					1	1	
v_0^j							

Ray of even length. The blocks of v .

	v_6^j	v_5^j	v_4^j	v_3^j	v_2^j	v_1^j	v_0^j
v_6^j	$\frac{R_0(\lambda)}{R_5(\lambda)}$	$\frac{\sqrt{R_4(\lambda)R_6(\lambda)}}{R_5(\lambda)}$					
v_5^j	$\frac{\sqrt{R_4(\lambda)R_6(\lambda)}}{R_5(\lambda)}$	$\frac{R_0(\lambda)}{R_5(\lambda)}$	1				
v_4^j		1	$\frac{R_0(\lambda)}{R_3(\lambda)}$	$\frac{\sqrt{R_2(\lambda)R_4(\lambda)}}{R_3(\lambda)}$			
v_3^j			$\frac{\sqrt{R_2(\lambda)R_4(\lambda)}}{R_3(\lambda)}$	$\frac{R_0(\lambda)}{R_3(\lambda)}$	1		
v_2^j				1	$\frac{R_0(\lambda)}{R_1(\lambda)}$	$\frac{\sqrt{R_0(\lambda)R_2(\lambda)}}{R_1(\lambda)}$	
v_1^j					$\frac{\sqrt{R_0(\lambda)R_2(\lambda)}}{R_1(\lambda)}$	$\frac{R_0(\lambda)}{R_1(\lambda)}$	
v_0^j							

Ray of odd length. The blocks of u .

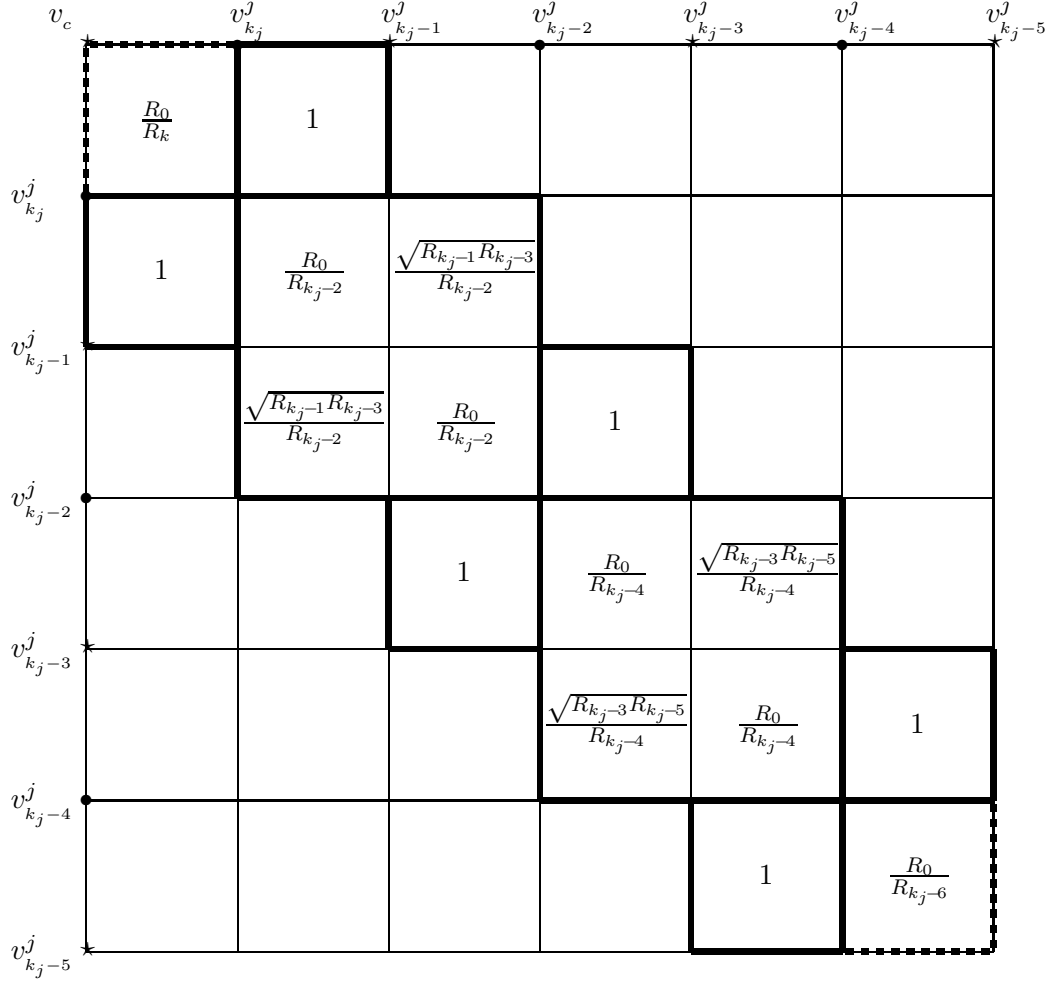
	v_6^j	v_5^j	v_4^j	v_3^j	v_2^j	v_1^j	v_0^j
v_6^j	$\frac{R_0(\lambda)}{R_5(\lambda)}$	$\frac{\sqrt{R_4(\lambda)R_6(\lambda)}}{R_5(\lambda)}$					
v_5^j	$\frac{\sqrt{R_4(\lambda)R_6(\lambda)}}{R_5(\lambda)}$	$\frac{R_0(\lambda)}{R_5(\lambda)}$	1				
v_4^j		1	$\frac{R_0(\lambda)}{R_3(\lambda)}$	$\frac{\sqrt{R_2(\lambda)R_4(\lambda)}}{R_3(\lambda)}$			
v_3^j			$\frac{\sqrt{R_2(\lambda)R_4(\lambda)}}{R_3(\lambda)}$	$\frac{R_0(\lambda)}{R_3(\lambda)}$	1		
v_2^j				1	$\frac{R_0(\lambda)}{R_1(\lambda)}$	$\frac{\sqrt{R_0(\lambda)R_2(\lambda)}}{R_1(\lambda)}$	
v_1^j					$\frac{\sqrt{R_0(\lambda)R_2(\lambda)}}{R_1(\lambda)}$	$\frac{R_0(\lambda)}{R_1(\lambda)}$	
v_0^j							

Ray of odd length. The blocks of v .

	v_6^j	v_5^j	v_4^j	v_3^j	v_2^j	v_1^j	v_0^j
v_6^j	$\frac{R_0(\lambda)}{R_6(\lambda)}$	1					
v_5^j	1	$\frac{R_0(\lambda)}{R_4(\lambda)}$	$\frac{\sqrt{R_3(\lambda)R_5(\lambda)}}{R_4(\lambda)}$				
v_4^j		$\frac{\sqrt{R_3(\lambda)R_5(\lambda)}}{R_4(\lambda)}$	$\frac{R_0(\lambda)}{R_4(\lambda)}$	1			
v_3^j			1	$\frac{R_0(\lambda)}{R_2(\lambda)}$	$\frac{\sqrt{R_1(\lambda)R_3(\lambda)}}{R_2(\lambda)}$		
v_2^j				$\frac{\sqrt{R_1(\lambda)R_3(\lambda)}}{R_2(\lambda)}$	$\frac{R_0(\lambda)}{R_2(\lambda)}$	1	
v_1^j					1	1	
v_0^j							

We will now look at a ray in the vicinity of the central vertex, but only deal with the cycles that do not “cross over” to another ray. For convenience the λ 's are omitted in the notation.

The blocks of u close to the center.



The blocks of v close to the center.

	v_c	$v_{k_j}^j$	$v_{k_j-1}^j$	$v_{k_j-2}^j$	$v_{k_j-3}^j$	$v_{k_j-4}^j$	$v_{k_j-5}^j$
	$\frac{R_0}{R_{k_j-1}}$	$\frac{\sqrt{R_{k_j} R_{k_j-2}}}{R_{k_j-1}}$					
$v_{k_j}^j$	$\frac{\sqrt{R_{k_j} R_{k_j-2}}}{R_{k_j-1}}$	$\frac{R_0}{R_{k_j-1}}$	1				
$v_{k_j-1}^j$		1	$\frac{R_0}{R_{k_j-3}}$	$\frac{\sqrt{R_{k_j-2} R_{k_j-4}}}{R_{k_j-3}}$			
$v_{k_j-2}^j$			$\frac{\sqrt{R_{k_j-2} R_{k_j-4}}}{R_{k_j-3}}$	$\frac{R_0}{R_{k_j-3}}$	1		
$v_{k_j-3}^j$				1	$\frac{R_0}{R_{k_j-5}}$	$\frac{\sqrt{R_{k_j-4} R_{k_j-6}}}{R_{k_j-5}}$	
$v_{k_j-4}^j$					$\frac{\sqrt{R_{k_j-4} R_{k_j-6}}}{R_{k_j-5}}$	$\frac{R_0}{R_{k_j-5}}$	
$v_{k_j-5}^j$							

We will now look at the cycles “crossing” the central vertex. We will list these cycles by the vertices involved. To indicate which vertices are considered fixed, we will underline the fixed ones. I.e the cycle $\underline{v}_1 v_2 \underline{v}_3 v_4$ is different from the cycle $v_1 \underline{v}_2 v_3 \underline{v}_4$. The first corresponds to an entry of u and the second to an entry of v .

For simplicity we will call the vertices $v_{k_j}^j$ for a_j , $j = 1, \dots, m$, and the vertex v_c will be called c .

The only cycles, for which we have not yet determined the modulus of the corresponding elements in u and v , span an $m \times m$ matrix of u , indexed by

$$\begin{pmatrix} \underline{c}a_1 \underline{c}a_1 & \underline{c}a_1 \underline{c}a_2 & \cdots & \underline{c}a_1 \underline{c}a_m \\ \underline{c}a_2 \underline{c}a_1 & \underline{c}a_2 \underline{c}a_2 & \cdots & \underline{c}a_2 \underline{c}a_m \\ \vdots & \vdots & \ddots & \vdots \\ \underline{c}a_m \underline{c}a_1 & \underline{c}a_m \underline{c}a_2 & \cdots & \underline{c}a_m \underline{c}a_m \end{pmatrix} \quad (2.6)$$

In the above we saw, that the modulus of the entry of u corresponding to the cycle $\underline{c}a_i \underline{c}a_i$ is $\frac{1}{R_{k_i}(\lambda)}$.

The cycles $\underline{c}a_i \underline{c}a_j$, $i \neq j$, are the only cycles fixing a_i and a_j , hence they must span a 1×1 block

of v and the modulus of the entry of u corresponding to the cycle $\underline{c}a_i\underline{c}a_j$, $i \neq j$, is

$$\sqrt{\frac{R_{k_i-1}(\lambda)R_{k_j-1}(\lambda)}{R_{k_i}(\lambda)R_{k_j}(\lambda)}}.$$

Hence the moduli of the elements of u corresponding to the cycles in (2.6) are given by

$$\begin{pmatrix} \frac{1}{R_{k_1}(\lambda)} & \sqrt{\frac{R_{k_1-1}(\lambda)R_{k_2-1}(\lambda)}{R_{k_1}(\lambda)R_{k_2}(\lambda)}} & \cdots & \sqrt{\frac{R_{k_1-1}(\lambda)R_{k_m-1}(\lambda)}{R_{k_1}(\lambda)R_{k_m}(\lambda)}} \\ \sqrt{\frac{R_{k_1-1}(\lambda)R_{k_2-1}(\lambda)}{R_{k_1}(\lambda)R_{k_2}(\lambda)}} & \frac{1}{R_{k_2}(\lambda)} & \cdots & \sqrt{\frac{R_{k_2-1}(\lambda)R_{k_m-1}(\lambda)}{R_{k_2}(\lambda)R_{k_m}(\lambda)}} \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{\frac{R_{k_1-1}(\lambda)R_{k_m-1}(\lambda)}{R_{k_1}(\lambda)R_{k_m}(\lambda)}} & \sqrt{\frac{R_{k_2-1}(\lambda)R_{k_m-1}(\lambda)}{R_{k_2}(\lambda)R_{k_m}(\lambda)}} & \cdots & \frac{1}{R_{k_m}(\lambda)} \end{pmatrix}.$$

To show that the matrix, formed by the squares of the entries of the matrix above, is doubly stochastic, we need to show

$$\frac{1}{R_{k_i}(\lambda)^2} + \sum_{j \neq i} \frac{R_{k_i-1}(\lambda)R_{k_j-1}(\lambda)}{R_{k_i}(\lambda)R_{k_j}(\lambda)} = 1.$$

Since $\sum_j \frac{R_{k_i-1}(\lambda)}{R_{k_i}(\lambda)} = \lambda$ (see 2.5), we have

$$\begin{aligned} & \frac{1}{R_{k_i}(\lambda)^2} + \sum_{j \neq i} \frac{R_{k_i-1}(\lambda)R_{k_j-1}(\lambda)}{R_{k_i}(\lambda)R_{k_j}(\lambda)} \\ &= \frac{1}{R_{k_i}(\lambda)^2} + \frac{R_{k_i-1}(\lambda)}{R_{k_i}(\lambda)} \left(\lambda - \frac{R_{k_i-1}(\lambda)}{R_{k_i}(\lambda)} \right) \\ &= \left(1 + \lambda R_{k_i-1}(\lambda)R_{k_i}(\lambda) - R_{k_i-1}(\lambda)R_{k_i-1}(\lambda) \right) / R_{k_i}(\lambda)^2 \\ &= \left(1 + R_{k_i-1}(\lambda) \left(\lambda R_{k_i}(\lambda) - R_{k_i-1}(\lambda) \right) \right) / R_{k_i}(\lambda)^2 \\ &= \left(1 + R_{k_i-1}(\lambda)R_{k_i+1}(\lambda) \right) / R_{k_i}(\lambda)^2. \end{aligned}$$

Hence we need to show

$$R_{k_i}(\lambda)^2 = 1 + R_{k_i-1}(\lambda)R_{k_i+1}(\lambda).$$

To show that the moduli squared of the 2×2 blocks of u and v form doubly stochastic matrices, we must show

$$R_j(\lambda)^2 = 1 + R_{j-1}(\lambda)R_{j+1}(\lambda),$$

which is the same identity as above. A proof of this identity is found as part of the proof of lemma 3.1.

Remark 2.9 If u is a $n \times n$ unitary and I denotes the $n \times n$ identity matrix, then the matrix

$$\begin{pmatrix} tu & \sqrt{1-t^2}I \\ \sqrt{1-t^2}I & -tu^* \end{pmatrix}, \quad 0 \leq t \leq 1$$

is unitary in $M_{2n}(\mathbb{C})$.

The only part of the bi-unitary condition which is non-trivial to solve, is the $M_m(M_n(\mathbb{C}))$ part indexed by

$$\left(\underline{ca}_{j_1} \underline{ca}_{j_2} \right)_{j_1, j_2=1}^m.$$

If we have a solution to this part, it will determine some of the entries in the 2×2 blocks of u resp. v . However, at most one entry of a 2×2 block is determined, and the three remaining entries can be determined by the above remark. Continuing the argument, as we move towards the end of a ray, we see that for each 2×2 block only one entry is determined by the previous blocks, and hence a solution to the entire 2×2 block can be determined.

If we put

$$\alpha_j = \frac{R_{k_j-1}(\lambda)}{R_{k_j}(\lambda)} \text{ and } \delta_j = \frac{1}{R_{k_j}(\lambda)}$$

we have

Proposition 2.10 *If $\Gamma = S(k_1, \dots, k_m)$ and α_j, δ_j are defined as above, there exists a commuting square of the form (2.1) if and only if there exists $n \times n$ unitaries u_{ij} such that*

$$\begin{pmatrix} \delta_1 u_{11} & \sqrt{\alpha_1 \alpha_2} u_{12} & \cdots & \sqrt{\alpha_1 \alpha_m} u_{1m} \\ \sqrt{\alpha_1 \alpha_2} u_{21} & \delta_2 u_{22} & \cdots & \sqrt{\alpha_2 \alpha_m} u_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{\alpha_m \alpha_1} u_{m1} & \sqrt{\alpha_m \alpha_2} u_{m2} & & \delta_m u_{mm} \end{pmatrix}$$

is a unitary matrix.

For $\Gamma = S(k_1, k_2, k_3, k_4)$ the answer to this problem is given by the theorems 4.11 and 4.14, which state, that if we label the δ 's such that $\delta_1 \geq \delta_2 \geq \delta_3 \geq \delta_4 > 0$, then

1. We need only look at $n = 2$, and a solution exists if and only if $\delta_1 - \delta_2 - \delta_3 - \delta_4 \leq 0$.
2. A solution in the case $n = 1$ exists if and only if $\delta_1 - \delta_2 - \delta_3 - \delta_4 \leq 0$ and $\delta_1 - \delta_2 - \delta_3 + \delta_4 \geq 0$.

Since the proof of the above results is long, we have devoted section 4 to the proof, and will concentrate on the 3-stars in the remainder of this section.

Lemma 2.11 *Let*

$$D = \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{pmatrix}$$

be doubly stochastic, and put $\alpha = d_{11} d_{21}$, $\beta = d_{12} d_{22}$ and $\gamma = d_{13} d_{23}$, then there exists a unitary $u = (u_{ij})$ such that $|u_{ij}|^2 = d_{ij}$, $i, j = 1, 2, 3$, if and only if $\sqrt{\alpha}$, $\sqrt{\beta}$ and $\sqrt{\gamma}$ satisfy the triangle inequality. The last condition is equivalent to

$$\alpha^2 + \beta^2 + \gamma^2 - 2\alpha\beta - 2\alpha\gamma - 2\beta\gamma \leq 0.$$

Proof If such a unitary exists, we can choose it to be of the form

$$\begin{pmatrix} \sqrt{d_{11}} & \sqrt{d_{12}} & \sqrt{d_{13}} \\ \sqrt{d_{21}} & \sqrt{d_{22}}e^{i\phi} & \sqrt{d_{23}}e^{i\theta} \\ \sqrt{d_{31}} & \sqrt{d_{32}}e^{i\psi} & \sqrt{d_{33}}e^{i\eta} \end{pmatrix}$$

and orthogonality of the two first rows implies

$$\sqrt{d_{11}d_{21}} + \sqrt{d_{12}d_{22}}e^{i\phi} + \sqrt{d_{13}d_{23}}e^{i\theta} = 0 \Leftrightarrow \sqrt{\alpha} + \sqrt{\beta}e^{i\phi} + \sqrt{\gamma}e^{i\theta} = 0. \quad (2.7)$$

Conversely, if we can find scalars $e^{i\psi}$ and $e^{i\theta}$ such that (2.7) is satisfied, the double stochastic property of D assures, that we can find a unitary of the desired form, by putting the 3'rd row equal to the conjugate vector product of the 1'st and 2'nd row.

Hence a necessary and sufficient condition for the existence of a unitary with the stated properties is

$$\begin{aligned} & \sqrt{\alpha}, \sqrt{\beta} \text{ and } \sqrt{\gamma} \text{ satisfy the triangle inequality} \\ \Leftrightarrow & |\sqrt{\alpha} - \sqrt{\beta}| \leq \sqrt{\gamma} \leq \sqrt{\alpha} + \sqrt{\beta} \\ \Leftrightarrow & \alpha^2 + \beta^2 + \gamma^2 - 2\alpha\beta - 2\alpha\gamma - 2\beta\gamma \leq 0 \end{aligned}$$

□

Lemma 2.12 *Let*

$$D = \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{pmatrix}$$

be doubly stochastic, and put $\alpha = d_{11}d_{21}$, $\beta = d_{12}d_{22}$ and $\gamma = d_{13}d_{23}$. If there exists $n \times n$ unitaries u_{ij} , $i, j = 1, 2, 3$, such that

$$\begin{pmatrix} \sqrt{d_{11}}u_{11} & \sqrt{d_{12}}u_{12} & \sqrt{d_{13}}u_{13} \\ \sqrt{d_{21}}u_{21} & \sqrt{d_{22}}u_{22} & \sqrt{d_{23}}u_{23} \\ \sqrt{d_{31}}u_{31} & \sqrt{d_{32}}u_{32} & \sqrt{d_{33}}u_{33} \end{pmatrix}$$

is unitary, then $\sqrt{\alpha}$, $\sqrt{\beta}$ and $\sqrt{\gamma}$ satisfy the triangle inequality.

Proof If u_{ij} exists, we have

$$\sqrt{d_{11}d_{21}}u_{11}^*u_{21} + \sqrt{d_{12}d_{22}}u_{12}^*u_{22} + \sqrt{d_{13}d_{23}}u_{13}^*u_{23} = 0 \Leftrightarrow \sqrt{\alpha}u_{11}^*u_{21} + \sqrt{\beta}u_{12}^*u_{22} + \sqrt{\gamma}u_{13}^*u_{23} = 0$$

Let $\|\cdot\|_{HS}$ denote the Hilbert-Schmidt norm on $M_n(\mathbb{C})$, then $\|u\|_{HS} = n$ for any unitary

$u \in M_n(\mathbb{C})$, and we have

$$\begin{aligned} & \|\sqrt{\alpha}u_{11}^*u_{21} + \sqrt{\beta}u_{12}^*u_{22} + \sqrt{\gamma}u_{13}^*u_{23}\|_{HS} = 0 \\ \Leftrightarrow & \|\sqrt{\alpha}u_{11}^*u_{21}\|_{HS}, \|\sqrt{\beta}u_{12}^*u_{22}\|_{HS}, \|\sqrt{\gamma}u_{13}^*u_{23}\|_{HS} \text{ satisfy the triangle inequality} \\ \Leftrightarrow & \sqrt{\alpha}, \sqrt{\beta} \text{ and } \sqrt{\gamma} \text{ satisfy the triangle inequality} \end{aligned}$$

□

Lemmas 2.11 and 2.12 now tell us, that we need only look at $n = 1$ in the diagram (2.1), when Γ is a 3-star.

Now let $\Gamma = S(k_1, k_2, k_3)$, the critical part of the bi-unitarity condition of (2.1) is the existence of a 3×3 unitary u such that

$$|u_{ij}|^2 = \begin{cases} \delta_i^2, & i = j \\ \alpha_i \alpha_j, & i \neq j \end{cases}$$

where $\delta_i = \frac{1}{R_{k_i}(\lambda)}$ and $\alpha_i = \frac{R_{k_i-1}(\lambda)}{R_{k_i}(\lambda)}$. Note that

$$\delta_i = \sqrt{\alpha_i^2 - \lambda \alpha_i + 1}, \quad i = 1, 2, 3, \quad (2.8)$$

and that the eigenvalue equation (2.5) is

$$\alpha_1 + \alpha_2 + \alpha_3 = \lambda.$$

Put $\alpha = \delta_1^2 \alpha_1 \alpha_2$, $\beta = \alpha_1 \alpha_2 \delta_2^2$ and $\gamma = \alpha_1 \alpha_2 \alpha_3^2$, then by lemma 2.11 u exists if and only if

$$\begin{aligned} \alpha^2 + \beta^2 + \gamma^2 - 2\alpha\beta - 2\alpha\gamma - 2\beta\gamma &\leq 0 \\ \Updownarrow \\ (\gamma - \alpha - \beta)^2 &\leq 4\alpha\beta \end{aligned},$$

and since

$$\delta_1^2 = \alpha_1^2 - \lambda \alpha_1 + 1 = \alpha_1^2 - (\alpha_1 + \alpha_2 + \alpha_3) \alpha_1 + 1 = 1 - \alpha_1 \alpha_2 - \alpha_1 \alpha_3,$$

and

$$\delta_2^2 = 1 - \alpha_1 \alpha_2 - \alpha_2 \alpha_3,$$

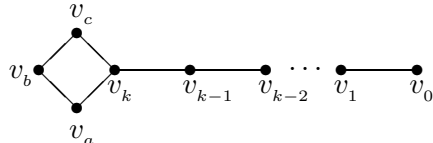
we get

$$\begin{aligned} &(\gamma - \alpha - \beta)^2 \leq 4\alpha\beta \\ \Updownarrow \\ &(\alpha_3^2 + 2\alpha_1 \alpha_2 + \alpha_1 \alpha_3 + \alpha_2 \alpha_3 - 2)^2 \leq 4(1 - \alpha_1 \alpha_2 - \alpha_1 \alpha_3)(1 - \alpha_1 \alpha_2 - \alpha_2 \alpha_3) \\ \Updownarrow \\ &\alpha_3^4 + \alpha_1^2 \alpha_3^2 + \alpha_2^2 \alpha_3^2 + 2\alpha_1 \alpha_3^3 + 2\alpha_2 \alpha_3^3 - 4\alpha_3^2 + 2\alpha_1 \alpha_2 \alpha_3^2 \leq 0 \\ \Updownarrow \\ &\alpha_3^2((\alpha_1 + \alpha_2 + \alpha_3)^2 - 4) \leq 0 \\ \Updownarrow \\ &\lambda = \alpha_1 + \alpha_2 + \alpha_3 \leq 2. \end{aligned}$$

Hence a 3-star Γ can only define the inclusions of a symmetric commuting square of the form (2.1) if $\|\Gamma\| \leq 2$.

We shall now briefly discuss the remaining type of graph, which may produce commuting squares of the form (2.1), with index in the interval (4, 5).

Consider the graph



If α_j denotes the coordinate of the Perron-Frobenius vector at the vertex v_j , the vector is given by:

$$\alpha_j = 2 \frac{R_j(\lambda)}{R_{k+1}(\lambda)}, \quad j = 0, \dots, k, \quad \alpha_a = 1, \quad \alpha_b = \frac{2}{\lambda}, \quad \alpha_c = 1.$$

The Perron-Frobenius eigenvalue, λ , is the largest solution to

$$\frac{2}{\lambda} + 2 \frac{R_k(\lambda)}{R_{k+1}(\lambda)} = \lambda,$$

which is also the Perron-Frobenius eigenvalue of $S(1, 1, k+1, k+1)$.

As we shall see all these 4-stars give rise to indices of irreducible subfactors of the hyperfinite II_1 -factor. We will therefore just mention, that our computations for $\text{kite}(k)$ show that all the graphs can form a commuting square of the form (2.1) for $n = 2$, but none can define such a commuting square if $n = 1$.

3 The 4-stars Satisfying the Conditions

In this section we shall determine which 4-stars satisfy the conditions of Theorems 4.11 and 4.14, i.e. if $1 \leq k_1 \leq k_2 \leq k_3 \leq k_4$, and λ is the Perron–Frobenius eigenvalue of $S(k_1, k_2, k_3, k_4)$, we put $\delta_i = \frac{1}{R_{k_i}(\lambda)}$, and the conditions are

$$(1) \quad \delta_1 - \delta_2 - \delta_3 - \delta_4 \leq 0 \text{ see (4.8)} \quad (3.1)$$

$$(2) \quad \delta_1 - \delta_2 - \delta_3 + \delta_4 \geq 0 \text{ see (4.9)} \quad (3.2)$$

The computations will use the following lemmas extensively:

Lemma 3.1 $\frac{R_{n-1}(\lambda)}{R_n(\lambda)}$ is increasing in n and decreasing in λ , when $\lambda \geq 2$.

Proof Consider

$$\begin{aligned} & R_n(\lambda)^2 - \lambda R_n(\lambda) R_{n-1}(\lambda) + R_{n-1}(\lambda)^2 \\ & - (R_n(\lambda)^2 - \lambda R_n(\lambda) R_{n+1}(\lambda) + R_{n+1}(\lambda)^2) \\ & = R_{n-1}(\lambda)^2 - \lambda R_n(\lambda) R_{n-1}(\lambda) - R_{n+1}(\lambda)^2 + \lambda R_n(\lambda) R_{n+1}(\lambda) \\ & = R_{n-1}(\lambda)(R_{n-1}(\lambda) - \lambda R_n(\lambda)) + R_{n+1}(\lambda)(\lambda R_n(\lambda) - R_{n+1}(\lambda)) \\ & = -R_{n-1}(\lambda) R_{n+1}(\lambda) + R_{n+1}(\lambda) R_{n-1}(\lambda) \\ & = 0. \end{aligned}$$

Hence $R_n(\lambda)^2 - \lambda R_n(\lambda) R_{n-1}(\lambda) + R_{n-1}(\lambda)^2$ is independent of n .

For $n = 1$ we have

$$R_1(\lambda)^2 - \lambda R_1(\lambda) R_0(\lambda) + R_0(\lambda)^2 = \lambda^2 - \lambda^2 + 1 = 1,$$

i.e. for all n and for all λ

$$R_n(\lambda)^2 - \lambda R_n(\lambda) R_{n-1}(\lambda) + R_{n-1}(\lambda)^2 = 1. \quad (3.3)$$

We can rewrite (3.3) as follows

$$\begin{aligned} 1 &= R_n(\lambda)^2 - \lambda R_n(\lambda) R_{n-1}(\lambda) + R_{n-1}(\lambda)^2 \\ &= R_n(\lambda)^2 + R_{n-1}(\lambda)(-\lambda R_n(\lambda) + R_{n-1}(\lambda)) \\ &= R_n(\lambda)^2 - R_{n-1}(\lambda) R_{n+1}(\lambda), \end{aligned}$$

and conclude

$$R_n(\lambda)^2 = R_{n-1}(\lambda) R_{n+1}(\lambda) + 1.$$

We now have

$$\frac{R_{n-1}(\lambda)}{R_n(\lambda)} \bigg/ \frac{R_n(\lambda)}{R_{n+1}(\lambda)} = \frac{R_{n-1}(\lambda) R_{n+1}(\lambda)}{R_n(\lambda)^2} < 1,$$

and hence $\frac{R_n(\lambda)}{R_{n+1}(\lambda)}$ is increasing in n . (Recall that $R_n(\lambda) > 0$ for all $\lambda \geq 2$.)

From the recursion formula, we have

$$\frac{R_{n+1}(\lambda)}{R_n(\lambda)} = \lambda - \frac{R_{n-1}(\lambda)}{R_n(\lambda)},$$

so if we can show that $\frac{R_{n-1}(\lambda)}{R_n(\lambda)}$ is decreasing in λ , we can conclude that also $\frac{R_n(\lambda)}{R_{n+1}(\lambda)}$ is decreasing in λ . For $n = 0$ we have $\frac{R_0(\lambda)}{R_1(\lambda)} = \frac{1}{\lambda}$, which clearly is decreasing in λ . \square

Corollary 3.2 For $m \geq 1$, $n \geq 0$, and $\lambda \geq 2$, $\frac{R_n(\lambda)}{R_{n+m}(\lambda)}$ has the following properties

1. $\frac{R_n(\lambda)}{R_{n+m}(\lambda)}$ is decreasing in λ .
2. $\frac{R_n(\lambda)}{R_{n+m}(\lambda)}$ is increasing in n .
3. $\frac{R_n(\lambda)}{R_{n+m}(\lambda)}$ is decreasing in m .

Proof

1 and 2.

$$\frac{R_n(\lambda)}{R_{n+m}(\lambda)} = \frac{R_n(\lambda)}{R_{n+1}(\lambda)} \frac{R_{n+1}(\lambda)}{R_{n+2}(\lambda)} \cdots \frac{R_{n+m-1}(\lambda)}{R_{n+m}(\lambda)},$$

and the right-hand side clearly has the stated properties.

3. For $\lambda \geq 2$ we can write $\lambda = e^x + e^{-x}$ for some $x \geq 0$. We have

$$\frac{R_n(\lambda)}{R_{n+1}(\lambda)} = \frac{e^{nx} - e^{-nx}}{e^{(n+1)x} - e^{-(n+1)x}} \underset{n \rightarrow \infty}{\nearrow} e^{-x} \leq 1,$$

and since $\frac{R_n(\lambda)}{R_{n+1}(\lambda)} \leq \frac{R_{n+1}(\lambda)}{R_{n+2}(\lambda)}$ we have

$$\frac{R_n(\lambda)}{R_{n+1}(\lambda)} \leq 1 \quad \text{for all } n.$$

We now have

$$\frac{R_n(\lambda)}{R_{n+m+1}(\lambda)} \bigg/ \frac{R_n(\lambda)}{R_{n+m}(\lambda)} = \frac{R_{n+m}(\lambda)}{R_{n+m+1}(\lambda)} \leq 1,$$

and hence

$$\frac{R_n(\lambda)}{R_{n+m+1}(\lambda)} \leq \frac{R_n(\lambda)}{R_{n+m}(\lambda)}.$$

\square

Remark 3.3 For $\lambda \geq 2$, $\lambda = e^x + e^{-x}$, $x \geq 0$ we have $R_n(\lambda) = \frac{e^{nx} - e^{-nx}}{e^x - e^{-x}}$, and hence

$$\lim_{n \rightarrow \infty} \frac{R_n(\lambda)}{R_{n+m}(\lambda)} = e^{-mx}.$$

The following lemma reduces the number of 4-stars for which we have to check condition (3.1).

Lemma 3.4 *Let λ be the Perron–Frobenius eigenvalue of $S(j, j + n_1, j + n_2, j + n_3)$, $j \geq 1$, $0 \leq n_1 \leq n_2 \leq n_3$, and let λ_1 be the Perron–Frobenius eigenvalue of $S(j, j + m_1, j + m_2, j + m_3)$, $j \geq 1$, $0 \leq m_1 \leq m_2 \leq m_3$. If $m_1 \leq n_1$, $m_2 \leq n_2$ and $m_3 \leq n_3$ then*

$$R_j(\lambda) \left(\frac{1}{R_{j+n_1}(\lambda)} + \frac{1}{R_{j+n_2}(\lambda)} + \frac{1}{R_{j+n_3}(\lambda)} \right) \leq R_j(\lambda_1) \left(\frac{1}{R_{j+m_1}(\lambda_1)} + \frac{1}{R_{j+m_2}(\lambda_1)} + \frac{1}{R_{j+m_3}(\lambda_1)} \right).$$

Proof Since $\lambda_1 \leq \lambda$ corollary 3.2 1. and 3. gives

$$\frac{R_j(\lambda)}{R_{j+n_i}(\lambda)} \leq \frac{R_j(\lambda)}{R_{j+m_i}(\lambda)} \leq \frac{R_j(\lambda_1)}{R_{j+m_i}(\lambda_1)}, \text{ for } i = 1, 2, 3.$$

This proves the statement. □

The following corollary is just a restatement of lemma 3.4.

Corollary 3.5

- (a) *If $S(j, j + n_1, j + n_2, j + n_3)$ satisfies condition (3.1), then $S(j, j + m_1, j + m_2, j + m_3)$ also satisfies condition (3.1), whenever $m_1 \leq n_1$, $m_2 \leq n_2$ and $m_3 \leq n_3$.*
- (b) *If $S(j, j + n_1, j + n_2, j + n_3)$ does not satisfy condition (3.1), then $S(j, j + m_1, j + m_2, j + m_3)$ does not satisfy condition (3.1), whenever $m_1 \geq n_1$, $m_2 \geq n_2$ and $m_3 \geq n_3$.*

Recall that the Perron–Frobenius eigenvalue of $S(i, j, k, l)$ satisfies the equation

$$\frac{R_{i-1}(\lambda)}{R_i(\lambda)} + \frac{R_{j-1}(\lambda)}{R_j(\lambda)} + \frac{R_{k-1}(\lambda)}{R_k(\lambda)} + \frac{R_{l-1}(\lambda)}{R_l(\lambda)} = \lambda$$

and that the polynomials $R_n(\lambda)$ are defined recursively by

$$R_0(\lambda) = 1, \quad R_1(\lambda) = \lambda, \quad R_{n+1}(\lambda) = \lambda R_n(\lambda) - R_{n-1}(\lambda)$$

Lemma 3.6

- 1. *If λ is the Perron–Frobenius eigenvalue of $S(j, j + 1, j + 1, j + 1)$, then $\frac{R_j(\lambda)}{R_{j+1}(\lambda)} = \frac{1}{\sqrt{3}}$.*
- 2. *If λ is the Perron–Frobenius eigenvalue of $S(j, j + 2, j + 2, j + 2)$, then $\frac{R_j(\lambda)}{R_{j+2}(\lambda)} = \frac{1}{3}$.*

Proof The recursion formula can be rewritten as

$$\lambda - \frac{R_{j-1}(\lambda)}{R_j(\lambda)} = \frac{R_{j+1}(\lambda)}{R_j(\lambda)}.$$

Using this we have

1. The eigenvalue equation is

$$\frac{R_{j-1}(\lambda)}{R_j(\lambda)} + 3\frac{R_j(\lambda)}{R_{j+1}(\lambda)} = \lambda \Leftrightarrow 3\left(\frac{R_j(\lambda)}{R_{j+1}(\lambda)}\right)^2 = 1 \Leftrightarrow \frac{R_j(\lambda)}{R_{j+1}(\lambda)} = \frac{1}{\sqrt{3}}.$$

2. The eigenvalue equation is

$$\frac{R_{j-1}(\lambda)}{R_j(\lambda)} + 3\frac{R_{j+1}(\lambda)}{R_{j+2}(\lambda)} = \lambda \Leftrightarrow 3\frac{R_j(\lambda)}{R_{j+2}(\lambda)} = 1.$$

□

Let λ_∞ be the Perron–Frobenius eigenvalue of $S(\infty, \infty, \infty, \infty)$, $\lambda_\infty = \frac{4}{\sqrt{3}}$. We shall first consider condition (3.1) which, for $\lambda =$ Perron–Frobenius eigenvalue of $S(k_1, k_2, k_3, k_4)$, $k_1 \leq k_2 \leq k_3 \leq k_4$ is equivalent to

$$R_{k_1}(\lambda) \left(\frac{1}{R_{k_2}(\lambda)} + \frac{1}{R_{k_3}(\lambda)} + \frac{1}{R_{k_4}(\lambda)} \right) \geq 1. \quad (3.4)$$

We divide the discussion into several steps:

(A) $S(j, j, k, l)$, $j \leq k \leq l$ trivially satisfy condition (3.4).

(B) Consider $S(j, j+1, j+1, j+m)$, $m \geq 1$, with Perron–Frobenius eigenvalue λ . By corollary 3.2 1. and 2. we obtain

$$2\frac{R_j(\lambda)}{R_{j+1}(\lambda)} + \frac{R_j(\lambda)}{R_{j+m}(\lambda)} \geq 2\frac{R_1(\lambda)}{R_2(\lambda)} \geq 2\frac{R_1(\lambda_\infty)}{R_2(\lambda_\infty)} = \frac{8\sqrt{3}}{13} > 1.$$

(C) Consider $S(j, j+1, j+2, j+4)$ with Perron–Frobenius eigenvalue λ .

For $j \geq 3$ we have

$$\text{RHS}(3.4) \geq R_3(\lambda_\infty) \left(\frac{1}{R_4(\lambda_\infty)} + \frac{1}{R_5(\lambda_\infty)} + \frac{1}{R_7(\lambda_\infty)} \right) \approx 1.01 > 1.$$

For $j = 2$ we have $\lambda \approx 2.2862 < \sqrt{5.25} = \lambda_0$, and hence

$$\text{RHS}(3.4) \geq R_2(\lambda_0) \left(\frac{1}{R_3(\lambda_0)} + \frac{1}{R_4(\lambda_0)} + \frac{1}{R_6(\lambda_0)} \right) \approx 1.02 > 1.$$

For $j = 1$ we have $\lambda \approx 2.2291 < \sqrt{5} = \lambda_0$, and hence

$$\text{RHS}(3.4) \geq R_1(\lambda_0) \left(\frac{1}{R_2(\lambda_0)} + \frac{1}{R_3(\lambda_0)} + \frac{1}{R_5(\lambda_0)} \right) = \frac{6\sqrt{5} + 11}{24} > 1.$$

I.e. $S(j, j+1, j+2, j+4)$ satisfies condition (3.1) for all j , and hence so does $S(j, j+1, j+2, j+2)$ and $S(j, j+1, j+2, j+3)$ by corollary 3.5.

(D) By lemma 3.6 2. it is easily seen that $S(j, j+2, j+2, j+2)$ satisfies condition (3.1) for all j .

To show that the 4-stars listed in (A)-(D) are the only ones that satisfy condition (3.1), corollary 3.5 tells us that we only need to prove that

(E) $S(j, j+2, j+2, j+3)$, $j \geq 1$,

(F) $S(j, j+1, j+3, j+3)$, $j \geq 1$,

(G) $S(j, j+1, j+2, j+5)$, $j \geq 1$

do not satisfy condition (1).

(E) Let λ be the Perron–Frobenius eigenvalue of $S(j, j+2, j+2, j+3)$, and let λ_1 be the Perron–Frobenius eigenvalue of $S(j, j+2, j+2, j+2)$. Since $\lambda_1 < \lambda$ we have

$$\text{RHS}(3.4) < 2 \frac{R_j(\lambda_1)}{R_{j+2}(\lambda_1)} + \frac{R_j(\lambda)}{R_{j+3}(\lambda)} < \frac{2}{3} + e^{-3x}.$$

The solution to $\frac{2}{3} + e^{-3x} < 1$ is $x > \frac{1}{3} \log 3$, corresponding to $\lambda > \lambda_2 = 2 \cosh(x) = 3^{\frac{1}{3}} + 3^{-\frac{1}{3}} \approx 2.1356$. Since the Perron–Frobenius eigenvalue of $S(1, 3, 3, 4) \approx 2.2411$, we obtain the statement of (E).

(F) Let λ be the Perron–Frobenius eigenvalue of $S(j, j+1, j+3, j+3)$ and λ_1 be the Perron–Frobenius eigenvalue of $S(j, j+1, j+1, j+1)$, then

$$\text{RHS}(3.4) < \frac{R_j(\lambda_1)}{R_{j+1}(\lambda_1)} \left(1 + 2 \frac{R_{j+1}(\lambda)}{R_{j+3}(\lambda)} \right) < \frac{1}{\sqrt{3}} (1 + 2e^{-2x}).$$

The solution to $\frac{1}{\sqrt{3}} (1 + 2e^{-2x}) < 1$ corresponds to $\lambda > \lambda_2 \approx 2.2579$. For $j = 2$ we have $\lambda \approx 2.2870 > \lambda_2$, so (F) is proved for $j \geq 2$.

In the case $j = 1$ we have $\lambda \approx 2.2323 > 2.22 = \lambda_3$, hence

$$\text{RHS}(3.4) < R_1(\lambda_3) \left(\frac{1}{R_2(\lambda_3)} + \frac{2}{R_4(\lambda_3)} \right) \approx 0.9878 < 1.$$

(G) Let λ be the Perron–Frobenius eigenvalue of $S(j, j+1, j+2, j+5)$ and λ_1 be the Perron–Frobenius eigenvalue of $S(j, j+1, j+1, j+1)$. Then $\text{RHS}(3.4) < \frac{1}{\sqrt{3}} (e^{-x} + e^{-4x})$, and the solution of $\frac{1}{\sqrt{3}} (e^{-x} + e^{-4x}) < 1$ corresponds to $\lambda > \lambda_2 \approx 2.0035$. If $j = 1$ we have $\lambda \approx 2.2298 > \lambda_2$, which proves (G).

By corollary 3.5 neither of the 4-stars listed below satisfy condition (3.1)

- (1) $S(j, j+2+n_1, j+2+n_2, j+3+n_3)$, $j \geq 1$, $0 \leq n_1 \leq n_2 \leq n_3$ (implied by (E))
- (2) $S(j, j+1+n_1, j+3+n_2, j+3+n_3)$, $j \geq 1$, $0 \leq n_1 \leq n_2 \leq n_3$ (implied by (F))
- (3) $S(j, j+1+n_1, j+2+n_2, j+5+n_3)$, $j \geq 1$, $0 \leq n_1 \leq n_2 \leq n_3$ (implied by (G))

Since we will only be concerned with condition (3.2) when condition (3.1) is satisfied, we will now determine which of the 4-stars listed in (A)-(D) satisfy condition (3.2).

If λ is the Perron–Frobenius eigenvalue of $S(j, j + n_1, j + n_2, j + n_3)$, $0 \leq n_1 \leq n_2 \leq n_3$, condition (3.2) is

$$R_j(\lambda) \left(\frac{1}{R_{j+n_1}(\lambda)} + \frac{1}{R_{j+n_2}(\lambda)} - \frac{1}{R_{j+n_3}(\lambda)} \right) \leq 1. \quad (3.5)$$

(A') $S(j, j, k, l)$ $j \leq k \leq l$ is easily seen to satisfy (3.5) if and only if $k = 1$.

(B') Put $n_1 = 1$, $n_2 = 1$ and $n_3 = m$.

If $m = 2$ or $m = 3$, we have

$$\text{RHS}(3.5) < \frac{1}{\sqrt{3}} \left(2 - \frac{R_2(\lambda_\infty)}{R_4(\lambda_\infty)} \right) \approx 0.9686 < 1.$$

Let $m \geq 4$. Then

$$\text{RHS}(3.5) \geq \frac{R_2(\lambda_\infty)}{R_3(\lambda_\infty)} (2 - e^{-3x})$$

and the solution to $\frac{R_2(\lambda_\infty)}{R_3(\lambda_\infty)} (2 - e^{-3x}) > 1$ corresponds to $\lambda > \lambda_2 \approx 2.2546$. As the Perron–Frobenius eigenvalue of $S(2, 3, 3, 6)$ is 2.2823(approx.), we have shown that $S(j, j + 1, j + 1, j + m)$, $j \geq 2$, $m \geq 4$, does not satisfy condition (3.2).

A similar argument shows that $S(j, j + 1, j + 1, j + m)$, $j \geq 1$, $m \geq 6$, does not satisfy condition (3.2), hence we now only need to consider $S(1, 2, 2, 5)$. We will show that (3.5) is satisfied with equality for $S(1, 2, 2, 5)$.

We have $R_5(\lambda) = R_1(\lambda)R_2(\lambda)(\lambda^2 - 3)$, and easy computations show

$$2 \frac{R_1(\lambda)}{R_2(\lambda)} - \frac{R_1(\lambda)}{R_5(\lambda)} = 1 \Leftrightarrow R_1(\lambda)R_2(\lambda)(\lambda - 2)(\lambda^3 - 4\lambda - 2) = 0,$$

and since $R_1(\lambda)R_2(\lambda)(\lambda - 2) \neq 0$, we must show $\lambda^3 - 4\lambda - 2 = 0$.

The eigenvalue equation is

$$\frac{R_0(\lambda)}{R_1(\lambda)} + 2 \frac{R_1(\lambda)}{R_2(\lambda)} + \frac{R_4(\lambda)}{R_5(\lambda)} = \lambda \Leftrightarrow \lambda(\lambda^2 - 1)(\lambda^3 - 4\lambda + 2)(\lambda^3 - 4\lambda - 2) = 0.$$

Since λ is the largest root of the above equation, we must have $\lambda^3 - 4\lambda - 2 = 0$.

(C') Put $n_1 = 1$, $n_2 = 2$, $n_3 = m$, then

$$\text{RHS}(3.5) < \frac{R_j(\lambda)}{R_{j+1}(\lambda)} \left(1 + \frac{R_{j+1}(\lambda)}{R_{j+2}(\lambda)} \right) < \frac{1}{\sqrt{3}} (1 + e^{-x}).$$

The solution of $\frac{1}{\sqrt{3}}(1 + e^{-x}) \leq 1$ corresponds to $\lambda > \lambda_2 \approx 2.0981$, and since the Perron–Frobenius eigenvalue of $S(1,2,3,3)$ is 2.2216 (approx.), $S(j, j+1, j+1, j+m)$ satisfies condition (3.2) for $j \geq 1$ and $m \geq 2$.

(D') Using lemma 3.6 2, $S(j, j+2, j+2, j+2)$ is easily seen to satisfy condition (3.2).

Hence the following is a total list of 4-stars which satisfy

Condition (3.1)

$S(j, j, k, l), 1 \leq j \leq k \leq l,$

$S(j, j+1, j+1, j+m), j \geq 1, m \geq 1,$

$S(j, j+1, j+2, j+m), j \geq 1, 2 \leq m \leq 4,$

$S(j, j+2, j+2, j+2), j \geq 1,$

Condition (3.1) & Condition (3.2)

$S(j, j, k, k), 1 \leq j \leq k,$

$S(j, j+1, j+1, j+m), j \geq 1, 1 \leq m \leq 3,$

$S(1, 2, 2, 5),$

$S(j, j+1, j+2, j+m), j \geq 1, 2 \leq m \leq 4,$

$S(j, j+2, j+2, j+2), j \geq 1.$

We end this section with a list of indices of irreducible subfactors of the hyperfinite II_1 -factor in the interval $(4, 5)$ which are produced by our construction. Most of the values are obtained by numerical methods, since it is not in general possible to solve the equation (2.5) for a 4-star analytically.

$S(1, 1, k, l), 1 \leq k \leq l$						
1	k	1	2	3	4	5
1		4.00000*				
2		4.30278	4.56155*			
3		4.41421	4.65109*	4.73205*		
4		4.46050	4.68554	4.76251	4.79129*	
5		4.48119	4.69963	4.77462	4.80262	4.81361*
6		4.49086	4.70559	4.77959	4.80721	4.81804
7		4.49551	4.70816	4.78165	4.80910	4.81986
8		4.49778	4.70928	4.78252	4.80988	4.82060
9		4.49889	4.70977	4.78289	4.81021	4.82092
10		4.49945	4.70998	4.78304	4.81035	4.82104
⋮		⋮	⋮	⋮	⋮	⋮
limit		4.50000	4.71015	4.78316	4.81044	4.82114

S(1, 1, k, l), 1 ≤ k ≤ l						
1	k	6	7	8	9	10
6		4.82240*				
7		4.82419	4.82596*			
8		4.82492	4.82668	4.82741*		
9		4.82522	4.82698	4.82771	4.82801*	
10		4.82535	4.82711	4.82783	4.82813	4.82825*
⋮		⋮	⋮	⋮	⋮	⋮
limit		4.82544	4.82720	4.82792	4.82822	4.82834

The limit values converge to $2 + 2\sqrt{2} \approx 4.82843$.

S(1, 2, 2, k)		S(1, 2, 3, k)	
k	λ^2	k	λ^2
2	4.79129*	3	4.93543*
3	4.86620*	4	4.95978*
4	4.89307*	5	4.96876*
5	4.90321*		
6	4.90715		
7	4.90869		
8	4.90930		
9	4.90955		
10	4.90964		
⋮	⋮		
limit	4.90971		

From S($j, j + 2, j + 2, j + 2$) there are no indices in the interval (4, 5).

By the discussion in the beginning of section 2, these are the only values which can arise from commuting squares of the form

$$\begin{array}{ccc} C & \subset_{nG^t} & D \\ \cup_G & & \cup_{G^t} \\ A & \subset_{nG} & A \end{array}, \quad n \in \mathbb{N}.$$

The lowest value is $\frac{1+\sqrt{13}}{2}$. The numbers marked with a star, are those which come from a commuting square of the form

$$\begin{array}{ccc} C & \subset_{G^t} & D \\ \cup_G & & \cup_{G^t} \\ A & \subset_G & A \end{array},$$

the lowest of which is $\frac{1+\sqrt{17}}{2}$.

We have, of course, a lot of other values of the index corresponding to the other 4-stars which satisfy the two conditions, but it would take up too much space to list some of the index values obtained from these graphs.

Also, as we will see in chapter II, all the limit values of the determined families are values of the index for an irreducible subfactor of the hyperfinite II_1 -factor.

4 Algebraic Necessities

Lemma 4.1 *Let $\delta_1 \geq \delta_2 \geq \delta_3 \geq \delta_4 \geq 0$. If $\delta_1 \pm \delta_2 \pm \delta_3 \pm \delta_4 = 0$ for some choice of signs, then either*

$$\delta_1 - \delta_2 - \delta_3 - \delta_4 = 0$$

or

$$\delta_1 - \delta_2 - \delta_3 + \delta_4 = 0.$$

Proof Assume that $f = \delta_1 - \delta_2 - \delta_3 - \delta_4 \neq 0$ and $g = \delta_1 - \delta_2 - \delta_3 + \delta_4 \neq 0$. We have

$$(\delta_1 - \delta_2) + (\delta_3 - \delta_4) \geq 0 \tag{4.1}$$

with equality if and only if $\delta_1 = \delta_2$ and $\delta_3 = \delta_4$. Thus equality in (4.1) implies $g = 0$. I.e. $\delta_1 - \delta_2 + \delta_3 - \delta_4 > 0$.

We also have

$$(\delta_1 - \delta_3) + (\delta_2 - \delta_4) \geq 0 \tag{4.2}$$

with equality if and only if $\delta_1 = \delta_2 = \delta_3 = \delta_4$. Hence equality in (4.2) implies $g = 0$. I.e. $\delta_1 + \delta_2 - \delta_3 - \delta_4 > 0$.

If at least two of $\delta_2, \delta_3, \delta_4$ must be chosen with positive sign, then either δ_2 or δ_3 is chosen positive. In this case we get: Sum of δ_i 's with signs \geq (4.1) resp. (4.2) > 0 . The above contradicts the possible choice of signs as stated. \square

Proposition 4.2 *Let $\delta_1 \geq \delta_2 \geq \delta_3 \geq \delta_4 \geq 0$. Then the following two conditions are equivalent:*

1. *There exists $t \in [0, \delta_4^2]$, and a choice of signs such that*

$$\sqrt{\delta_1^2 - t} \pm \sqrt{\delta_2^2 - t} \pm \sqrt{\delta_3^2 - t} \pm \sqrt{\delta_4^2 - t} = 0.$$

2. *$\delta_1 - \delta_2 - \delta_3 - \delta_4 \leq 0$ and $\delta_1 - \delta_2 - \delta_3 + \delta_4 \geq 0$.*

Proof Put $\delta_i(t) = \sqrt{\delta_i^2 - t}$, $i = 1, 2, 3, 4$, $t \in [0, \delta_4^2]$, then $\delta_1(t) \geq \delta_2(t) \geq \delta_3(t) \geq \delta_4(t) \geq 0$. Put

$$f(t) = \delta_1(t) - \delta_2(t) - \delta_3(t) - \delta_4(t)$$

and

$$g(t) = \delta_1(t) - \delta_2(t) - \delta_3(t) + \delta_4(t).$$

By lemma 4.1 1 is equivalent to 1': $f(t) = 0$ for some $t \in [0, \delta_4^2]$ or $g(t) = 0$ for some $t \in [0, \delta_4^2]$. We will first prove that $2 \Rightarrow 1'$.

In the above notation the statement of 2 is $f(0) \leq 0$ and $g(0) \geq 0$, and since $f(\delta_4^2) = g(\delta_4^2)$ there exists a $t \in [0, \delta_4^2]$, such that $f(t) = 0$ or $g(t) = 0$.

Proof of 1' \Rightarrow 2: Assume 1'. If $\delta_4 = 0$, the only possible value of t is $t = 0$, so 2 follows immediately. Let $\delta_4 > 0$ and assume that 2 is false, i.e.

(a) $\delta_1 - \delta_2 - \delta_3 - \delta_4 > 0$ or

(b) $\delta_1 - \delta_2 - \delta_3 + \delta_4 < 0$.

If (a) is valid, we get

$$f'(t) = \frac{1}{2}(-\delta_1(t)^{-1} + \delta_2(t)^{-1} + \delta_3(t)^{-1} + \delta_4(t)^{-1}), \quad 0 \leq t < \delta_4^2,$$

and $\delta_1(0) \geq \delta_2(0) \geq \delta_3(0) \geq \delta_4(0) > 0$ implies $f'(0) > 0$. Assume that there exists a $t \in (0, \delta_4^2)$ such that $f'(t) = 0$, and let t_0 be the smallest such t . Since $f'(0) > 0$ we get $f'(t) > 0$, $0 \leq t < t_0$, and hence $f(t_0) \geq f(0) > 0$, i.e. $\delta_1(t_0) - \delta_2(t_0) - \delta_3(t_0) - \delta_4(t_0) > 0$. The above argument (in the case $t = t_0$ instead of $t = 0$) yields $f'(t_0) > 0$, which is a contradiction. I.e. $f'(t) > 0$, for all $t \in [0, \delta_4^2)$ and thus $f(t) \geq f(0) > 0$, for all $t \in [0, \delta_4^2]$. As $g(t) = f(t) + 2\delta_4(t)$ we also get $g(t) > 0$, for all $t \in [0, \delta_4^2]$. This proves the implication in case (a).

Assume (b). For all $t \in [0, \delta_4^2]$

$$\begin{aligned} g'(t) &= \frac{1}{2}(-\delta_1(t)^{-1} + \delta_2(t)^{-1} + \delta_3(t)^{-1} - \delta_4(t)^{-1}) \\ &= \frac{1}{2} \left(\frac{\delta_1(t) - \delta_2(t)}{\delta_1(t)\delta_2(t)} - \frac{\delta_3(t) - \delta_4(t)}{\delta_3(t)\delta_4(t)} \right). \end{aligned}$$

Since $\delta_1(t) \geq \delta_3(t)$ and $\delta_2(t) \geq \delta_4(t)$ we have $\delta_1(t)\delta_2(t) \geq \delta_3(t)\delta_4(t)$, and we get

$$g'(t) \leq \frac{1}{2\delta_1(t)\delta_2(t)}g(t), \quad \text{for all } t \in [0, \delta_4^2]$$

in particular $g'(0) < 0$. Assume there exists $t \in (0, \delta_4^2)$ such that $g'(t) = 0$, and let t_0 be the smallest such t . Then $g(t)$ is decreasing on $[0, t_0]$, and hence $g(t_0) \leq g(0) < 0$, but then we get $g'(t_0) < 0$, which is a contradiction. I.e. $g(t) \leq g(0) < 0$, for all $t \in [0, \delta_4^2]$, and since $f(t) = g(t) - 2\delta_4(t)$ we get $f(t) < 0$, for all $t \in [0, \delta_4^2]$. This proves the implication in case (b). \square

For the rest of this section we let $\alpha_1, \alpha_2, \alpha_3$ and α_4 denote positive reals, such that we for $\lambda = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ have

$$\lambda > 2 \quad \text{and} \quad 0 < \alpha_i \leq \frac{\lambda - \sqrt{\lambda^2 - 4}}{2}, \quad i = 1, 2, 3, 4,$$

and we put

$$\delta_i = \sqrt{\alpha_i^2 - \lambda\alpha_i + 1}, \quad i = 1, 3, 2, 4.$$

This is well defined, because $\frac{1}{2}(\lambda - \sqrt{\lambda^2 - 4})$ is the smallest root of the polynomial $x^2 - \lambda x + 1$.

Remark 4.3 Since for $\{i, j, k, l\} = \{1, 2, 3, 4\}$

$$\delta_i^2 + \alpha_i\alpha_j + \alpha_i\alpha_k + \alpha_i\alpha_l = (\alpha_i + \alpha_j + \alpha_k + \alpha_l)\alpha_i - \lambda\alpha_i + 1 = 1$$

the matrix

$$D = \begin{pmatrix} \delta_1^2 & \alpha_1 \alpha_2 & \alpha_1 \alpha_3 & \alpha_1 \alpha_4 \\ \alpha_1 \alpha_2 & \delta_2^2 & \alpha_2 \alpha_3 & \alpha_2 \alpha_4 \\ \alpha_1 \alpha_3 & \alpha_2 \alpha_3 & \delta_3^2 & \alpha_3 \alpha_4 \\ \alpha_1 \alpha_4 & \alpha_2 \alpha_4 & \alpha_3 \alpha_4 & \delta_4^2 \end{pmatrix} \quad (4.3)$$

is doubly stochastic, i.e. rows and columns have sum 1.

Lemma 4.4 *With $\alpha_1, \alpha_2, \alpha_3$ and α_4 as above:*

1. $\delta_i < \frac{\lambda}{2} - \alpha_i, i = 1, 2, 3, 4.$
2. $\delta_i + \delta_j < \alpha_k + \alpha_l, \text{ when } \{i, j, k, l\} = \{1, 2, 3, 4\}.$
3. $|\delta_i - \delta_j| \geq \alpha_i - \alpha_j, \text{ with equality if and only if } \alpha_i = \alpha_j.$

Proof

1. Since $\lambda > 2, \alpha_i^2 - \lambda \alpha_i + 1 < (\alpha_i - \frac{\lambda}{2})^2$. Hence

$$\delta_i < |\alpha_i - \frac{\lambda}{2}| = \frac{\lambda}{2} - \alpha_i.$$

2. Let (i, j, k, l) be a permutation of $(1, 2, 3, 4)$. By 1

$$\delta_i + \delta_j < \lambda - \alpha_i - \alpha_j = \alpha_l + \alpha_k.$$

3. The function

$$f(\alpha) = \sqrt{\alpha^2 - \lambda \alpha + 1}, \quad 0 < \alpha \leq \frac{\lambda - \sqrt{\lambda^2 - 4}}{2},$$

is strictly decreasing, and

$$f'(\alpha) = -\frac{\frac{\lambda}{2} - \alpha}{\sqrt{\alpha^2 - \lambda \alpha + 1}} < -1, \quad \left(\alpha \neq \frac{\lambda - \sqrt{\lambda^2 - 4}}{2} \right),$$

because $(\frac{\lambda}{2} - \alpha)^2 > \alpha^2 - \lambda \alpha + 1$, since $\lambda > 2$. Hence

$$|f(\alpha) - f(\beta)| > |\alpha - \beta|$$

for all $\alpha, \beta \in [0, \frac{1}{2}(\lambda - \sqrt{\lambda^2 - 4})]$, $\alpha \neq \beta$. This proves 3. □

The rest of this section will be taken up by a study of the properties of some special matrices, which eventually will lead to the main results of this section.

Lemma 4.5 *Let $u \in M_4(M_n(\mathbb{C}))$, $u = (u_{ij})_{i,j=1}^4$ be a unitary matrix such that $u_{ij} u_{ij}^* = \alpha_{ij} I_n$, $i, j = 1, \dots, 4$, where $(\alpha_{ij})_{i,j=1}^4$ is doubly stochastic and symmetric, then*

$$\sqrt{\alpha_{i_1 i_1}} \leq \sqrt{\alpha_{i_2 i_2}} + \sqrt{\alpha_{i_3 i_3}} + \sqrt{\alpha_{i_4 i_4}}, \quad \{i_1, i_2, i_3, i_4\} = \{1, 2, 3, 4\}.$$

Proof $u = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a, b, c, d \in M_2(M_n(\mathbb{C}))$. Since u is unitary, $a^*a = 1 - c^*c$ and $dd^* = 1 - cc^*$. Therefore a^*a and dd^* have the same list of eigenvalues, so

$$\|a\|_2^2 = \text{Tr}(a^*a) = \text{Tr}(dd^*) = \|d\|_2^2$$

and

$$|\det(a)| = \det(a^*a)^{\frac{1}{2}} = \det(dd^*)^{\frac{1}{2}} = |\det(d)|.$$

By the assumptions

$$u_{ij} = \sqrt{\alpha_{ij}} V_{ij}, \quad i, j = 1, 2, 3, 4$$

where the V'_{ij} s are unitary $n \times n$ matrices.

Set $V = V_{21}^* V_{22} V_{12}^* V_{11}$. Then

$$a = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = \begin{pmatrix} V_{11} & 0 \\ 0 & V_{22} \end{pmatrix} \begin{pmatrix} \sqrt{\alpha_{11}} & \sqrt{\alpha_{12}} \\ \sqrt{\alpha_{21}} & \sqrt{\alpha_{22}} V \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & V_{11}^* V_{12} \end{pmatrix}.$$

Hence

$$|\det(a)| = \left| \det \begin{pmatrix} \sqrt{\alpha_{11}} & \sqrt{\alpha_{12}} \\ \sqrt{\alpha_{21}} & \sqrt{\alpha_{22}} V \end{pmatrix} \right|.$$

Sine V is unitary, it is unitary equivalent to a diagonal matrix, with diagonal elements v_1, \dots, v_n of modulus 1. Hence

$$|\det(a)| = \prod_{i=1}^n \left| \det \begin{pmatrix} \sqrt{\alpha_{11}} & \sqrt{\alpha_{12}} \\ \sqrt{\alpha_{21}} & \sqrt{\alpha_{22}} v_i \end{pmatrix} \right|,$$

which shows that

$$|\sqrt{\alpha_{11}\alpha_{22}} - \alpha_{12}|^n \leq |\det(a)| \leq (\sqrt{\alpha_{11}\alpha_{22}} + \alpha_{12})^n.$$

Similarly

$$|\sqrt{\alpha_{33}\alpha_{44}} - \alpha_{34}|^n \leq |\det(d)| \leq (\sqrt{\alpha_{33}\alpha_{44}} + \alpha_{34})^n.$$

Since $|\det(a)| = |\det(d)|$ it follows that the two intervals

$$I_1^0 = [|\sqrt{\alpha_{11}\alpha_{22}} - \alpha_{12}|, \sqrt{\alpha_{11}\alpha_{22}} + \alpha_{12}],$$

$$I_2^0 = [|\sqrt{\alpha_{33}\alpha_{44}} - \alpha_{34}|, \sqrt{\alpha_{33}\alpha_{44}} + \alpha_{34}]$$

have non-empty intersection. Hence also $I_1 \cap I_2 \neq \emptyset$, where I_1 and I_2 are the two (possibly larger) intervals

$$I_1 = [\alpha_{12} - \sqrt{\alpha_{11}\alpha_{22}}, \alpha_{12} + \sqrt{\alpha_{11}\alpha_{22}}],$$

$$I_2 = [\alpha_{34} - \sqrt{\alpha_{33}\alpha_{44}}, \alpha_{34} + \sqrt{\alpha_{33}\alpha_{44}}].$$

Since $\alpha_{11} + \alpha_{22} + 2\alpha_{12} = \frac{1}{n}\|a\|_2^2 = \frac{1}{n}\|d\|_2^2 = \alpha_{33} + \alpha_{44} + 2\alpha_{34}$ the two intervals

$$J_1 = \alpha_{11} + \alpha_{22} + 2\alpha_{12} - 2I_1 = [(\sqrt{\alpha_{11}} - \sqrt{\alpha_{22}})^2, (\sqrt{\alpha_{11}} + \sqrt{\alpha_{22}})^2]$$

$$J_2 = \alpha_{33} + \alpha_{44} + 2\alpha_{34} - 2I_2 = [(\sqrt{\alpha_{33}} - \sqrt{\alpha_{44}})^2, (\sqrt{\alpha_{33}} + \sqrt{\alpha_{44}})^2]$$

also intersect. Hence

$$[|\sqrt{\alpha_{11}} - \sqrt{\alpha_{22}}|, \sqrt{\alpha_{11}} + \sqrt{\alpha_{22}}] \cap [|\sqrt{\alpha_{33}} - \sqrt{\alpha_{44}}|, \sqrt{\alpha_{33}} + \sqrt{\alpha_{44}}] \neq \emptyset,$$

which is equivalent to the “four-angle” inequality for $\sqrt{\alpha_{11}}$, $\sqrt{\alpha_{22}}$, $\sqrt{\alpha_{33}}$ and $\sqrt{\alpha_{44}}$, stated in the lemma. \square

Corollary 4.6 *If $u \in M_4(\mathbb{C})$ is unitary and $|u_{ij}| = |u_{ji}|$, $i, j = 1, 2, 3, 4$, then*

$$|u_{ii}| \leq |u_{jj}| + |u_{kk}| + |u_{ll}|, \quad \{i, j, k, l\} = \{1, 2, 3, 4\}.$$

Proof Set $n = 1$ in lemma 4.5. □

For the rest of this section D denotes the double stochastic matrix with entries

$$D_{ij} = \begin{cases} \delta_i & \text{if } i = j \\ \sqrt{\alpha_i \alpha_j} & \text{if } i \neq j \end{cases}$$

we then have the following

Lemma 4.7 *If $u \in M_4(\mathbb{C})$ is a unitary, such that $|u_{ij}| = D_{ij}$, $i, j = 1, 2, 3, 4$, then $u \neq u^t$.*

Proof Suppose $u = u^t$. By exchanging u with wuw for a suitably chosen diagonal unitary operator, w , we can obtain $u_{12}, u_{13}, u_{14} \geq 0$. So u is of the form

$$\begin{pmatrix} d_1 & \sqrt{\alpha_1 \alpha_2} & \sqrt{\alpha_1 \alpha_3} & \sqrt{\alpha_1 \alpha_4} \\ \sqrt{\alpha_1 \alpha_2} & d_2 & \sqrt{\alpha_2 \alpha_3} \sigma & \sqrt{\alpha_2 \alpha_4} \rho \\ \sqrt{\alpha_1 \alpha_3} & \sqrt{\alpha_2 \alpha_3} \sigma & d_3 & \sqrt{\alpha_3 \alpha_4} \tau \\ \sqrt{\alpha_1 \alpha_4} & \sqrt{\alpha_2 \alpha_4} \rho & \sqrt{\alpha_3 \alpha_4} \tau & d_4 \end{pmatrix}$$

where $\rho, \sigma, \tau \in \mathbb{C}$, $|\rho| = |\sigma| = |\tau| = 1$ and $|d_i| = \delta_i$, $i = 1, 2, 3, 4$. Orthogonality implies

$$\begin{aligned} (1) \quad \bar{d}_1 + d_2 + \alpha_3 \sigma + \alpha_4 \rho &= 0 \\ (2) \quad d_1 + \bar{d}_3 + \alpha_2 \bar{\sigma} + \alpha_4 \bar{\tau} &= 0 \\ (3) \quad \bar{d}_2 \sigma + d_3 \bar{\sigma} + \alpha_1 + \alpha_4 \bar{\rho} \tau &= 0 \end{aligned}$$

which is equivalent to

$$\begin{aligned} (1') \quad \bar{d}_1 \bar{\sigma} + d_2 \bar{\sigma} + \alpha_3 + \alpha_4 \rho \bar{\sigma} &= 0 \\ (2') \quad d_1 \sigma + \bar{d}_3 \sigma + \alpha_2 + \alpha_4 \bar{\tau} \sigma &= 0 \\ (3') \quad \bar{d}_2 \sigma + d_3 \bar{\sigma} + \alpha_1 + \alpha_4 \bar{\rho} \tau &= 0 \end{aligned}$$

Hence the sum of the left-hand sides of (1'), (2') and (3') is 0, which implies

$$2\text{Re}(d_1 \sigma + \bar{d}_2 \sigma + \bar{d}_3 \sigma) + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 (\rho \bar{\sigma} + \bar{\tau} \sigma + \bar{\rho} \sigma) = 0 \Rightarrow \text{Im}(\rho \bar{\sigma} + \bar{\tau} \sigma + \bar{\rho} \sigma) = 0.$$

Let T be the triangle with vertices at ρ, σ, τ . Then

$$\text{area}(T) = \frac{1}{2} |\text{Im}((\tau - \sigma)(\bar{\rho} - \bar{\sigma}))| = \frac{1}{2} |\text{Im}(\rho \bar{\sigma} + \bar{\tau} \sigma + \bar{\rho} \sigma)| = 0.$$

Hence τ, σ and ρ lie on a straight line, and since $|\rho| = |\sigma| = |\tau| = 1$, at least two are equal.

Take for instance the case $\sigma = \rho$. In this case (1) states $\bar{d}_1 + d_2 + (\alpha_3 + \alpha_4) \sigma = 0 \Rightarrow$

$$\delta_1 + \delta_2 \geq |\bar{d}_1 + d_2| = \alpha_3 + \alpha_4.$$

Which contradicts lemma 4.4 2.

The other cases are treated similarly. □

Lemma 4.8 *Let a, b, c be $n \times n$ matrices, such that $aa^* + bb^* = 1$ and $a^*a + c^*c = 1$. Then $|\det(b)| = |\det(c)|$. If b is invertible there is one and only one $n \times n$ matrix d such that*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a unitary matrix, and d is given by

$$d = -(c^*)^{-1}a^*b = -ca^*(b^*)^{-1}.$$

Proof Since a^*a and aa^* have the same list of eigenvalues

$$|\det(b)|^2 = \det(1 - aa^*) = \det(1 - a^*a) = |\det(c)|^2.$$

Assume now $|\det(b)| = |\det(c)| \neq 0$. If

$$u = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is unitary, then $a^*b + c^*d = 0$ and $ca^* + db^* = 0$, hence

$$d = -(c^*)^{-1}a^*b \text{ and } d = -ca^*(b^*)^{-1}.$$

This proves uniqueness of d , and the stated formulas for d .

To prove existence, set $d = -(c^*)^{-1}a^*b$. Then $a^*b + c^*d = 0$. By the assumptions $a^*a + c^*c = 1$. Moreover, since $af(a^*a) = f(aa^*)a$ for any function f on $\text{sp}(a^*a) = \text{sp}(aa^*)$, we get

$$\begin{aligned} b^*b + d^*d &= b^*(1 + a(c^*c)^{-1}a^*)b \\ &= b^*(1 + a(1 - a^*a)^{-1}a^*)b \\ &= b^*(1 + (1 - aa^*)^{-1}aa^*)b \\ &= b^*(1 - aa^*)^{-1}b \\ &= b^*(bb^*)^{-1}b \\ &= 1. \end{aligned}$$

Hence $u^*u = 1$, i.e. u is unitary. □

Proposition 4.9 *If there exists a choice of signs such that $\delta_1 \pm \delta_2 \pm \delta_3 \pm \delta_4 = 0$, then there exists a selfadjoint unitary 4×4 -matrix u , with $\text{Tr}(u) = 0$ and $|u_{ij}| = D_{ij}$.*

Proof If $\delta_1 = \delta_2 = \delta_3 = \delta_4 = 0$, then all the α 's are equal, and since $\sum_i \alpha_i = \lambda$,

$$\alpha_i = \frac{\lambda}{4}, \quad i = 1, 2, 3, 4.$$

Since $0 = \delta_i^2 = \alpha_i^2 - \lambda\alpha_i + 1$, it follows that $\lambda = \frac{4}{\sqrt{3}}$, and thus

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \frac{1}{\sqrt{3}}.$$

In this case

$$u = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & i & -i \\ 1 & -i & 0 & i \\ 1 & i & -i & 0 \end{pmatrix}$$

is a selfadjoint unitary matrix with $\text{Tr}(u) = 0$, for which $|u_{ij}|^2 = D_{ij}$.

Assume now, that not all the δ'_i 's are 0. By the assumption we can choose $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \in \mathbb{R}$, such that $|\epsilon_i| = \delta_i$ and

$$\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 = 0.$$

The three numbers $\epsilon_1 + \epsilon_2$, $\epsilon_1 + \epsilon_3$ and $\epsilon_2 + \epsilon_3$ cannot all be zero, because this would imply $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0$ and $\epsilon_4 = -(\epsilon_1 + \epsilon_2 + \epsilon_3) = 0$, which contradicts that $\delta_i \neq 0$ for some i . Hence, by permuting the indices, we can obtain $\epsilon_1 + \epsilon_2 \neq 0$. This implies that $\epsilon_3 + \epsilon_4 = -(\epsilon_1 + \epsilon_2) \neq 0$.

We seek a solution of the form

$$u = \begin{pmatrix} \epsilon_1 & \sqrt{\alpha_1 \alpha_2} & \sqrt{\alpha_1 \alpha_3} & \sqrt{\alpha_1 \alpha_4} \\ \sqrt{\alpha_1 \alpha_2} & \epsilon_2 & \sqrt{\alpha_2 \alpha_3} \sigma & \sqrt{\alpha_2 \alpha_4} \bar{\tau} \\ \sqrt{\alpha_1 \alpha_3} & \sqrt{\alpha_2 \alpha_3} \sigma & * & * \\ \sqrt{\alpha_1 \alpha_4} & \sqrt{\alpha_2 \alpha_4} \tau & * & * \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

$a, b, c, d \in M_2(\mathbb{C})$, and $|\sigma| = |\tau| = 1$. Orthogonality of the 1'st and 2'nd column is equivalent to

$$\epsilon_1 + \epsilon_2 + \alpha_3 \sigma + \alpha_4 \tau = 0 \tag{4.4}$$

By lemma 4.4 2,

$$|\epsilon_1 + \epsilon_2| \leq |\epsilon_1| + |\epsilon_2| \leq \delta_1 + \delta_2 < \alpha_3 + \alpha_4,$$

and since $\epsilon_1 + \epsilon_2 = -(\epsilon_3 + \epsilon_4)$ lemma 4.4 3 gives

$$|\epsilon_1 + \epsilon_2| \geq ||\epsilon_3| - |\epsilon_4|| = |\delta_3 - \delta_4| \geq \alpha_3 - \alpha_4.$$

Hence $|\epsilon_3 + \epsilon_4|$, α_3 and α_4 satisfy the triangle inequality, so we can choose σ and $\tau \in \mathbb{C}$, $|\sigma| = |\tau| = 1$, such that (4.4) holds. Moreover $|\epsilon_1 + \epsilon_2| < \alpha_3 + \alpha_4$ implies that $\sigma \neq \tau$. Therefore the matrix

$$c = \begin{pmatrix} \sqrt{\alpha_1 \alpha_3} & \sqrt{\alpha_2 \alpha_3} \sigma \\ \sqrt{\alpha_1 \alpha_4} & \sqrt{\alpha_2 \alpha_4} \tau \end{pmatrix}$$

is invertible. By construction $a^*a + c^*c = 1$, and since $a = a^*$ and $b = c^*$, also $aa^* + bb^* = 1$.

Let

$$d = -(c^*)^{-1}a^*b = -b^{-1}ab$$

as in lemma 4.8. Since $bb^* = 1 - a^2$,

$$d = b^{-1}ab = -b^*(bb^*)^{-1}ab = -b^*(1 - a^2)^{-1}ab.$$

Hence $d = d^*$, i.e.

$$d = \begin{pmatrix} d_1 & z \\ \bar{z} & d_2 \end{pmatrix}, \quad d_1, d_2 \in \mathbb{R}, \quad z \in \mathbb{C}.$$

Moreover

$$\text{Tr}(d) = -\text{Tr}(b^{-1}ab) = -\text{Tr}(a),$$

i.e.

$$d_1 + d_2 = -(\epsilon_1 + \epsilon_2) = \epsilon_3 + \epsilon_4. \quad (4.5)$$

Since $u = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is unitary by lemma 4.8,

$$\alpha_1\alpha_3 + \alpha_2\alpha_3 + d_1^2 + |z|^2 = 1,$$

$$\alpha_1\alpha_4 + \alpha_2\alpha_4 + |z|^2 + d_2^2 = 1,$$

and since the matrix

$$D_{ij} = \begin{cases} \delta_i^2, & i = j \\ \alpha_i\alpha_j, & i \neq j \end{cases}$$

is doubly stochastic, it follows that

$$\begin{aligned} d_1^2 + |z|^2 &= \delta_3^2 + \alpha_3\alpha_4 \\ |z|^2 + d_2^2 &= \alpha_3\alpha_4 + \delta_4^2. \end{aligned} \quad (4.6)$$

Hence

$$(d_1 + d_2)(d_1 - d_2) = d_1^2 - d_2^2 = \delta_3^2 - \delta_4^2 = \epsilon_3^2 - \epsilon_4^2 = (\epsilon_3 + \epsilon_4)(\epsilon_3 - \epsilon_4).$$

But $d_1 + d_2 = \epsilon_3 + \epsilon_4 \neq 0$. Thus

$$d_1 - d_2 = \epsilon_3 - \epsilon_4, \quad (4.7)$$

so by (4.5) and (4.7), $d_1 = \epsilon_3$ and $d_2 = \epsilon_4$. Finally (4.6) gives $|z|^2 = \alpha_3\alpha_4$. Hence

$$u = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a selfadjoint unitary with

$$\text{Tr}(u) = \text{Tr}(a) + \text{Tr}(d) = 0,$$

and

$$|u_{ij}|^2 = D_{ij}, \quad i, j = 1, 2, 3, 4.$$

□

Proposition 4.10 *If there exists a unitary $u \in M_4(\mathbb{C})$ such that $|u_{ij}| = D_{ij}$, then u can be chosen as $u = \sqrt{1 - \gamma^2}v + i\gamma 1$, where v is selfadjoint unitary with $\text{Tr}(v) = 0$ and $\gamma \in [-1, 1]$.*

Proof By proposition 4.9 we may assume that there is no choice of signs such that $\delta_1 \pm \delta_2 \pm \delta_3 \pm \delta_4 = 0$. Particularly not all δ 's have the same value. If we relabel the δ 's to get $\delta_1 > \delta_2$, u can be chosen to be

$$\begin{pmatrix} \delta_1 & \sqrt{\alpha_1\alpha_2} & \sqrt{\alpha_1\alpha_3} & \sqrt{\alpha_1\alpha_4} \\ \sqrt{\alpha_1\alpha_2} & u_{22} & * & * \\ \sqrt{\alpha_1\alpha_3} & * & * & * \\ \sqrt{\alpha_1\alpha_4} & * & * & * \end{pmatrix}, \quad \text{where } |u_{22}| = \delta_2.$$

If $u = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is unitary, then so is $\begin{pmatrix} ae^{i\theta} & b \\ c & de^{-i\theta} \end{pmatrix}$. Hence we may substitute $e^{i\theta}\delta_1$ for δ_1 and $e^{-i\theta}u_{22}$ for u_{22} , where θ is chosen such that $e^{i\theta}(\delta_1 + \overline{u_{22}}) > 0$.

Put $u'_{11} = e^{i\theta}\delta_1$ and $u'_{22} = e^{-i\theta}u_{22}$, then $\text{Im}(u'_{11}) = \text{Im}(u'_{22})$ and u is now transformed to

$$\begin{pmatrix} \epsilon_1 + i\gamma & \sqrt{\alpha_1\alpha_2} & \sqrt{\alpha_1\alpha_3} & \sqrt{\alpha_1\alpha_4} \\ \sqrt{\alpha_1\alpha_2} & \epsilon_2 + i\gamma & \sqrt{\alpha_2\alpha_3}\mu & \sqrt{\alpha_2\alpha_4}\mu' \\ \sqrt{\alpha_1\alpha_3} & \sqrt{\alpha_2\alpha_3}\sigma & & \\ \sqrt{\alpha_1\alpha_4} & \sqrt{\alpha_2\alpha_4}\sigma' & & d \end{pmatrix},$$

where $\epsilon_1, \epsilon_2 \in \mathbb{R}$.

Orthogonality of the two first rows, respectively columns, gives

$$\begin{aligned} (1) \quad & \epsilon_1 + \epsilon_2 + \alpha_3\sigma + \alpha_4\sigma' = 0 \\ (2) \quad & \epsilon_1 + \epsilon_2 + \alpha_3\mu + \alpha_4\mu' = 0 \end{aligned}$$

Since $\epsilon_1 + \epsilon_2 > 0$ there are only two solutions to (1), hence either $\sigma = \mu$ (and $\sigma' = \mu'$) or $\sigma = \overline{\mu}$ (and $\sigma' = \overline{\mu'}$).

By lemma 4.4 $\epsilon_1 + \epsilon_2 \leq \delta_1 + \delta_2 < \alpha_3 + \alpha_4$, so the triangle $\begin{array}{c} \alpha_3\sigma \quad \alpha_4\sigma' \\ \triangle \\ \epsilon_1 + \epsilon_2 \end{array}$ does not degenerate to a straight line. I.e. σ and σ' have non-trivial imaginary parts.

We are now in one of the following situations

$$(a) \quad u = \begin{pmatrix} a + i\gamma 1 & b \\ b^t & d \end{pmatrix} \quad (b) \quad u = \begin{pmatrix} a + i\gamma 1 & b \\ b^* & d \end{pmatrix}$$

In case (a) d is uniquely determined as

$$d = -(\overline{b})^{-1}(a - i\gamma 1)b = -b^t(a - i\gamma 1)(b^*)^{-1},$$

hence $d = d^t \Rightarrow u = u^t$, which contradicts lemma 4.7. I.e. we must be in case (b).

Here we get $d = -(b^{-1})(a - i\gamma 1)b = -b^*(a - i\gamma 1)(b^*)^{-1}$, that is $d = d_{sa} + i\gamma 1$, where d_{sa} is selfadjoint. Moreover

$$\text{Tr}(\text{selfadjoint part of } u) = 0,$$

because

$$\text{Tr}(d_{sa}) = \text{Tr}(-b^{-1}ab) = -\text{Tr}(a).$$

Hence $u = s + i\gamma 1$, where s is selfadjoint with $\text{Tr}(s) = 0$. But $u^*u = 1$ implies $s^*s = (1 - \gamma^2)1$. Thus $|\gamma| \leq 1$ and $s = \sqrt{1 - \gamma^2}v$ for a selfadjoint unitary v with trace 0. \square

Theorem 4.11 *Let $\lambda, \delta_1 \geq \delta_2 \geq \delta_3 \geq \delta_4 > 0$ be defined by $0 < \alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \alpha_4$ as before. Then the following are equivalent*

1. There exists a unitary $u \in M_4(\mathbb{C})$, $u = (u_{ij})$ such that $|u_{ij}| = D_{ij}$.

2.

$$\delta_1 - \delta_2 - \delta_3 - \delta_4 \leq 0 \quad (4.8)$$

$$\delta_1 - \delta_2 - \delta_3 + \delta_4 \geq 0. \quad (4.9)$$

Proof $1 \Rightarrow 2$. If all the δ 's are equal, 2 is trivially fulfilled. If the δ 's are not all equal, proposition 4.10 states that u can be chosen as $u = v + i\gamma 1$, where v is selfadjoint $\text{Tr}(v) = 0$ and $\gamma \in \mathbb{R}$, i.e. $u_{kk} = \epsilon_k + i\gamma$ and $\sum_k \epsilon_k = 0$. Hence

$$\epsilon_k = \pm \sqrt{|u_{kk}|^2 - \gamma^2} = \pm \sqrt{\delta_k^2 - \gamma^2},$$

where $\gamma \leq \min\{\delta_1^2, \delta_2^2, \delta_3^2, \delta_4^2\} = \delta_4^2$, and proposition 4.2 gives the implication.

$2 \Rightarrow 1$. By proposition 4.2 we can choose $t \in [0, \delta_4^2]$ and signs such that

$$\sqrt{\delta_1^2 - t} \pm \sqrt{\delta_2^2 - t} \pm \sqrt{\delta_3^2 - t} \pm \sqrt{\delta_4^2 - t} = 0.$$

Put $a_i = \frac{1}{\sqrt{1-t}}\alpha_i$, then $\lambda_1 = \sum_i a_i = \frac{1}{\sqrt{1-t}}\lambda > 2$, and $a_i^2 - \lambda_1 a_i + 1 = \frac{\delta_i^2 - t}{1-t} \geq 0$

Put $d_i = \sqrt{a_i^2 - \lambda_1 a_i + 1} = \frac{1}{\sqrt{1-t}}\sqrt{\delta_i^2 - t}$. Since $\frac{1}{\lambda} \leq \frac{\alpha_i}{1-t}$ we get $\frac{1}{\lambda_1} \leq a_i$.

Moreover, since $a_i^2 - \lambda_1 a_i + 1 \geq 0$, either

$$a_i \leq \frac{\lambda_1 - \sqrt{\lambda_1^2 - 4}}{2} \quad \text{or} \quad a_i \geq \frac{\lambda_1 + \sqrt{\lambda_1^2 - 4}}{2}.$$

However $\alpha_i \leq \frac{\lambda - \sqrt{\lambda^2 - 4}}{2} \leq \frac{\lambda}{2}$ implies that $a_i \leq \frac{\lambda_1}{2}$. Hence $a_i \leq \frac{\lambda_1 - \sqrt{\lambda_1^2 - 4}}{2}$.

Proposition 4.9 now produces a selfadjoint unitary v , with $\text{Tr}(v) = 0$ such that

$$|v_{ij}|^2 = \begin{cases} d_i^2 & i = j \\ a_i a_j & i \neq j \end{cases}$$

Put $u = \sqrt{1-t}v + i\sqrt{t}1$, then u is unitary and

$$\begin{aligned} |u_{ii}|^2 &= (1-t)v_{ii} + t = (1-t)d_i^2 + t = \delta_i^2 \\ |u_{ij}|^2 &= (1-t)v_{ij} = (1-t)a_i a_j = \alpha_i \alpha_j \quad i \neq j \end{aligned}$$

□

Remark 4.12 Let $\delta_1 \geq \delta_2 \geq \delta_3 \geq \delta_4 \geq 0$. By trivial manipulations

1.

$$\begin{aligned} & \delta_1 - \delta_2 - \delta_3 - \delta_4 \leq 0 \\ \Updownarrow & \\ & (-\delta_1 + \delta_2 + \delta_3 + \delta_4)(\delta_1 - \delta_2 + \delta_3 + \delta_4)(\delta_1 + \delta_2 - \delta_3 + \delta_4)(\delta_1 + \delta_2 + \delta_3 - \delta_4) \leq 0 \end{aligned}$$

2.

$$\begin{aligned} & \delta_1 - \delta_2 - \delta_3 + \delta_4 \geq 0 \\ \Updownarrow & \\ & (\delta_1 + \delta_2 + \delta_3 + \delta_4)(\delta_1 - \delta_2 - \delta_3 + \delta_4)(\delta_1 - \delta_2 + \delta_3 - \delta_4)(\delta_1 + \delta_2 - \delta_3 - \delta_4) \geq 0. \end{aligned}$$

Hence theorem 4.11 can be stated in the symmetric form:

There exists a unitary $u \in M_4(\mathbb{C})$, $u = (u_{ij})$ such that $|u_{ij}| = D_{ij}$ if and only if

$$-\delta_1 \delta_2 \delta_3 \delta_4 \leq \delta_1^4 + \delta_2^4 + \delta_3^4 + \delta_4^4 - 2 \sum_{i < j} \delta_i \delta_j \leq \delta_1 \delta_2 \delta_3 \delta_4.$$

Proposition 4.13 *If $\delta_1, \delta_2, \delta_3, \delta_4$ satisfy the “four-angle” inequality*

$$\delta_i \leq \delta_j + \delta_k + \delta_l, \quad \{i, j, k, l\} = \{1, 2, 3, 4\}.$$

Then there is a unitary 4×4 matrix, u , with entries in the quaternions \mathbb{H} , such that

$$|u_{ij}|^2 = \begin{cases} \alpha_i \alpha_j, & i \neq j \\ \delta_i^2 & i = j. \end{cases} \quad (4.10)$$

Proof The quaternions $\mathbb{H} = \{a_1 + ia_2 + ja_3 + ka_4 \mid a_1, a_2, a_3, a_4 \in \mathbb{R}\}$ can be identified with the real subalgebra of $M_2(\mathbb{C})$ given by

$$\left\{ \begin{pmatrix} a_1 + ia_2 & a_2 + ia_3 \\ -a_2 + ia_3 & a_1 - ia_2 \end{pmatrix} \mid a_1, a_2, a_3, a_4 \in \mathbb{R} \right\}.$$

Therefore any $v \in M_n(\mathbb{H})$ can be considered as an element in $M_{2n}(\mathbb{C})$. Hence the second part of lemma 4.8 extends trivially to matrices with quaternionic entries.

By permuting the indices, we can assume that $\delta_1 + \delta_2 \leq \delta_3 + \delta_4$, so by the assumptions on $\delta_1, \delta_2, \delta_3$ and δ_4

$$|\delta_3 - \delta_4| \leq \delta_1 + \delta_2 \leq \delta_3 + \delta_4 \quad (4.11)$$

i.e. $\delta_1 + \delta_2, \delta_3$ and δ_4 satisfy the triangle inequality.

If $\delta_2 + \delta_2 = 0$, then $\delta_1 = \delta_2 = 0$ and $\delta_3 = \delta_4$. In this case (4.10) has a solution in $M_4(\mathbb{C}) \subset M_4(\mathbb{H})$ by proposition 4.9.

Hence we may assume that $\delta_1 + \delta_2 > 0$.

We seek a solution of the form

$$u = \begin{pmatrix} \delta_1 & \sqrt{\alpha_1 \alpha_2} & \sqrt{\alpha_1 \alpha_3} & \sqrt{\alpha_1 \alpha_4} \\ \sqrt{\alpha_1 \alpha_2} & \delta_2 & \sqrt{\alpha_2 \alpha_3} \sigma' & \sqrt{\alpha_2 \alpha_4} \tau' \\ \sqrt{\alpha_1 \alpha_3} & \sqrt{\alpha_2 \alpha_3} \sigma & * & * \\ \sqrt{\alpha_1 \alpha_4} & \sqrt{\alpha_2 \alpha_4} \tau & * & * \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad (4.12)$$

where $a, b, c, d \in M_2(\mathbb{H})$ and σ, σ', τ and τ' are quaternions of modulus 1. Orthogonality of the first two columns resp. rows is equivalent to

$$\delta_1 + \delta_2 + \alpha_3 \sigma + \alpha_4 \tau = 0, \quad (4.13)$$

resp.

$$\delta_1 + \delta_2 + \alpha_3 \sigma' + \alpha_4 \tau' = 0. \quad (4.14)$$

By lemma 4.4 2, 3 and (4.11)

$$|\alpha_3 - \alpha_4| \leq \delta_1 + \delta_2 < \alpha_3 + \alpha_4,$$

i.e. $\delta_1 + \delta_2, \alpha_3$ and α_4 satisfy the triangle inequality.

If σ, τ are unit quaternions satisfying (4.13), then

$$|\delta_1 + \delta_2 + \alpha_3 \sigma|^2 = \alpha_4^2.$$

Thus $\text{Re} \sigma = h$, where

$$h = \frac{\alpha_4^2 - \alpha_3^2 - (\delta_1 + \delta_2)^2}{2\alpha_3(\delta_1 + \delta_2)}.$$

Since $\delta_1 + \delta_2, \alpha_3$ and α_4 satisfy the triangle inequality, there are solutions to (4.14), so in particular $|h| \leq 1$.

Let (i, j, k) be the standard basis for the imaginary part of \mathbb{H} . Set

$$\sigma = h + i\sqrt{1-h^2} \quad \text{and} \quad \sigma' = h - (i \cos \theta + j \sin \theta)\sqrt{1-h^2}, \quad \theta \in [0, \pi]$$

Then $|\sigma| = |\sigma'| = 1$, $\text{Re}(\sigma) = \text{Re}(\sigma') = h$ and the angle between the imaginary parts of σ and σ' is $\pi - \theta$. In particular

$$\sigma' = \bar{\sigma} \quad \text{iff} \quad \theta = 0,$$

$$\sigma' = \sigma \quad \text{iff} \quad \theta = \pi.$$

Since $\text{Re}(\sigma) = \text{Re}(\sigma') = h$,

$$|\delta_1 + \delta_2 + \alpha_3 \sigma|^2 = |\delta_1 + \delta_2 + \alpha_3 \sigma'|^2 = \alpha_4^2.$$

Hence

$$\tau = \frac{1}{\alpha_4}(\delta_1 + \delta_2 + \alpha_3 \sigma)$$

and

$$\tau' = \frac{1}{\alpha_4}(\delta_1 + \delta_2 + \alpha_3 \sigma')$$

are unit quaternions, and (4.13) and (4.14) hold, i.e.

$$a^*a + c^*c = 1, \quad aa^* + bb^* = 1.$$

Moreover b and c are invertible, because the inequality $\delta_1 + \delta_2 < \alpha_3 + \alpha_4$ implies that $\sigma \neq \tau$ and $\sigma' \neq \tau'$. Thus, by lemma 4.8 and the remarks in the beginning of this proof,

$$d = -(c^*)^{-1}a^*b = -ca^*(b^*)^{-1}$$

defines a matrix in $M_2(\mathbb{H})$, such that

$$u = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is unitary.

Since $(|u_{ij}|^2)_{i,j}$ and $(D_{ij})_{i,j}$ are doubly stochastic matrices, which coincide on the two first rows and the two first columns

$$\begin{aligned} |d_{11}|^2 + |d_{12}|^2 &= D_{33} + D_{34} \\ |d_{21}|^2 + |d_{22}|^2 &= D_{43} + D_{44} \\ |d_{11}|^2 + |d_{21}|^2 &= D_{33} + D_{43} \\ |d_{12}|^2 + |d_{22}|^2 &= D_{34} + D_{44}. \end{aligned}$$

Since $D_{33} = \delta_3, D_{44} = \delta_4$ and $D_{34} = \alpha_3\alpha_4$ it follows that

$$\begin{pmatrix} |d_{11}|^2 & |d_{12}|^2 \\ |d_{21}|^2 & |d_{22}|^2 \end{pmatrix} = \begin{pmatrix} \delta_3^2 - \kappa & \alpha_3\alpha_4 + \kappa \\ \alpha_3\alpha_4 + \kappa & \delta_4^2 - \kappa \end{pmatrix} \quad (4.15)$$

for some constant, $\kappa = \kappa(\theta)$, depending on θ . Hence to prove the proposition we have to show, that $\theta \in [0, \pi]$ can be chosen such that $\kappa(\theta) = 0$.

If $\theta = 0$, then $b = c^*$, so as in the proof of proposition 4.9. we have $d = d^*$ and

$$\text{Tr}(d) = -\text{Tr}(a) = -(\delta_1 + \delta_2).$$

Thus $d_{11}, d_{22} \in \mathbb{R}$,

$$d_{11} + d_{22} = -(\delta_1 + \delta_2) \quad (4.16)$$

and by (4.15)

$$d_{11}^2 - d_{22}^2 = \delta_3^2 - \delta_4^2$$

Hence

$$d_{11} - d_{22} = -\frac{\delta_3^2 - \delta_4^2}{\delta_1 + \delta_2}. \quad (4.17)$$

By (4.16) and (4.17)

$$\begin{aligned} d_{11} &= -\frac{1}{2} \left((\delta_1 + \delta_2) + \frac{\delta_3^2 - \delta_4^2}{\delta_1 + \delta_2} \right), \\ d_{22} &= -\frac{1}{2} \left((\delta_1 + \delta_2) - \frac{\delta_3^2 - \delta_4^2}{\delta_1 + \delta_2} \right). \end{aligned}$$

Thus

$$\begin{aligned}\kappa(0) &= \frac{1}{2}(\delta_3^2 + \delta_4^2 - d_{11}^2 - d_{22}^2) \\ &= -\frac{1}{4} \left((\delta_1 + \delta_2)^2 - 2(\delta_3^2 + \delta_4^2) + \frac{(\delta_3^2 - \delta_4^2)^2}{(\delta_1 + \delta_2)^2} \right)\end{aligned}$$

Since the roots of the polynomial

$$t^2 - 2(\delta_3^2 + \delta_4^2)t + (\delta_3^2 - \delta_4^2)^2$$

are $(\delta_3 + \delta_4)^2$ and $(\delta_3 - \delta_4)^2$ it follows that

$$\kappa(0) = \frac{1}{4(\delta_1 + \delta_2)^2} ((\delta_3 + \delta_4)^2 - (\delta_1 + \delta_2)^2) ((\delta_1 + \delta_2)^2 - (\delta_3 - \delta_4)^2). \quad (4.18)$$

Note that $\kappa(0) \geq 0$, because $\delta_1 + \delta_2, \delta_3$ and δ_4 satisfy the triangle inequality (4.11).

Next we show that $\kappa(\pi) \leq 0$. Let $\theta \in [0, \pi]$. Since $\sigma, \tau \in \{a_1 + ia_2 \mid a_1, a_2 \in \mathbb{R}\} \cong \mathbb{C}$ one gets

$$(c^*)^{-1} = \frac{1}{\sqrt{\alpha_1 \alpha_2 \alpha_3 \alpha_4} (\bar{\tau} - \bar{\sigma})} \begin{pmatrix} \sqrt{\alpha_2 \alpha_4} \bar{\tau} & -\sqrt{\alpha_3 \alpha_4} \\ -\sqrt{\alpha_2 \alpha_3} \bar{\sigma} & \sqrt{\alpha_1 \alpha_3} \end{pmatrix}.$$

Hence

$$d_{21} = ((c^*)^{-1} ab)_{21} = \frac{\sqrt{\alpha_3}}{\sqrt{\alpha_4} (\bar{\tau} - \bar{\sigma})} (\alpha_1 - \delta_1 \bar{\sigma} + \delta_2 \sigma' - \alpha_2 \bar{\sigma} \sigma').$$

Since $\text{Re}(\sigma) = \text{Re}(\sigma') = h$ and

$$\text{Re}(\bar{\sigma} \sigma') = h^2 - (1 - h^2) \cos \theta, \quad \text{Re}(\sigma \sigma') = h^2 + (1 - h^2) \cos \theta,$$

we have

$$|\alpha_1 - \alpha_2 \bar{\sigma} \sigma' - \delta_1 \bar{\sigma} + \delta_2 \sigma'|^2 = \beta - \gamma \cos \theta,$$

where

$$\beta = \alpha_1^2 + \alpha_2^2 + \delta_1^2 + \delta_2^2 - 2h(\alpha_1 - \alpha_2)(\delta_1 - \delta_2) - 2h^2(\alpha_1 \alpha_2 + \delta_1 \delta_2)$$

and

$$\gamma = 2(1 - h^2)(\alpha_1 \alpha_2 - \delta_1 \delta_2).$$

Therefore

$$\kappa(\theta) = |d_{21}|^2 - \alpha_3 \alpha_4 = \frac{\alpha_3}{\alpha_4 |\bar{\tau} - \bar{\sigma}|^2} (\beta - \gamma \cos \theta) - \alpha_3 \alpha_4.$$

In particular

$$\kappa(0) - \kappa(\pi) = \frac{2\alpha_3 \gamma}{\alpha_4 |\bar{\tau} - \bar{\sigma}|^2} = \frac{4\alpha_3 (1 - h^2)}{\alpha_4 |\bar{\tau} - \bar{\sigma}|^2} (\alpha_1 \alpha_2 - \delta_1 \delta_2).$$

By (4.13)

$$(\delta_1 + \delta_2) \bar{\sigma} + \alpha_3 + \alpha_4 \tau \bar{\sigma} = 0.$$

Therefore

$$(\delta_1 + \delta_2) |\text{Im}(\sigma)| = \alpha_4 |\text{Im}(\tau \bar{\sigma})|,$$

from which

$$1 - h^2 = |\text{Im}(\sigma)|^2 = \frac{\alpha_4^2}{(\delta_1 + \delta_2)^2} |\text{Im}(\tau \bar{\sigma})|^2 = \frac{\alpha_4^2}{(\delta_1 + \delta_2)^2} (1 - \text{Re}(\tau \bar{\sigma}))^2.$$

Moreover $|\overline{\sigma} - \overline{\tau}|^2 = 2(1 - \operatorname{Re}(\tau\overline{\sigma}))$, hence

$$\kappa(0) - \kappa(\pi) = \frac{2\alpha_3\alpha_4(1 + \operatorname{Re}(\tau\overline{\sigma}))}{(\delta_1 + \delta_2)^2}(\alpha_1\alpha_2 - \delta_1\delta_2).$$

Again using (4.13)

$$|\alpha_3\sigma + \alpha_4\tau|^2 = (\delta_1 + \delta_2)^2.$$

Thus

$$\operatorname{Re}(\tau\overline{\sigma}) = \frac{(\delta_1 + \delta_2)^2 - \alpha_3^2 - \alpha_4^2}{2\alpha_3\alpha_4},$$

and

$$1 + \operatorname{Re}(\tau\overline{\sigma}) = \frac{(\delta_1 + \delta_2)^2 - (\alpha_3 - \alpha_4)^2}{2\alpha_3\alpha_4}.$$

All together

$$\kappa(0) - \kappa(\pi) = \frac{(\alpha_1\alpha_2 - \delta_1\delta_2)((\delta_1 + \delta_2)^2 - (\alpha_3 - \alpha_4)^2)}{(\delta_1 + \delta_2)^2} \quad (4.19)$$

From (4.18) and (4.19) it follows that $\kappa(\pi) \leq 0$ if and only if

$$(\alpha_1\alpha_2 - \delta_1\delta_2)((\delta_1 + \delta_2)^2 - (\alpha_3 - \alpha_4)^2) \geq \frac{1}{4}((\delta_3 + \delta_4)^2 - (\delta_1 + \delta_2)^2)((\delta_1 + \delta_2)^2 - (\delta_3 - \delta_4)^2). \quad (4.20)$$

By lemma 4.4 1,

$$\begin{aligned} \alpha_1\alpha_2 - \delta_1\delta_2 &> \frac{\lambda}{2}(\alpha_1 + \alpha_2) - \frac{\lambda^2}{4} \\ \alpha_3\alpha_4 - \delta_3\delta_4 &> \frac{\lambda}{2}(\alpha_3 + \alpha_4) - \frac{\lambda^2}{4}, \end{aligned}$$

hence

$$(\alpha_1\alpha_2 - \delta_1\delta_2) + (\alpha_3\alpha_4 - \delta_3\delta_4) > 0. \quad (4.21)$$

Since the matrix $D = (D_{ij})$ is doubly stochastic

$$D_{11} + D_{12} + D_{21} + D_{22} = D_{33} + D_{34} + D_{43} + D_{44},$$

i.e.

$$\delta_1^2 + \delta_2^2 + 2\alpha_1\alpha_2 = \delta_3^2 + \delta_4^2 + 2\alpha_3\alpha_4.$$

Equivalently

$$(\alpha_1\alpha_2 - \delta_1\delta_2) - (\alpha_3\alpha_4 - \delta_3\delta_4) = \frac{1}{2}((\delta_3 + \delta_4)^2 - (\delta_1 + \delta_2)^2), \quad (4.22)$$

so by adding (4.21) and (4.22),

$$(\alpha_1\alpha_2 - \delta_1\delta_2) > \frac{1}{4}((\delta_3 + \delta_4)^2 - (\delta_1 + \delta_2)^2).$$

Moreover, by lemma 4.4 3

$$(\delta_1 + \delta_2)^2 - (\alpha_3 - \alpha_4)^2 \geq (\delta_1 + \delta_2)^2 - (\delta_3 - \delta_4)^2.$$

This proves (4.20), because both factors on the right-hand side of (4.20) are non-negative. Hence

$$\kappa(0) \geq 0 \quad \text{and} \quad \kappa(\pi) \leq 0,$$

so $\kappa(\theta) = 0$ for some $\theta \in [0, \pi]$. This completes the proof. \square

Theorem 4.14 *Let $\delta_1 \geq \delta_2 \geq \delta_3 \geq \delta_4 > 0$ and λ be defined by $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ in the usual way. Then the following are equivalent*

1. *There exists $n \in \mathbb{N}$ and unitaries $v_{ij} \in M_n(\mathbb{C})$ such that $(D_{ij}v_{ij})_{i,j=1}^4$ is unitary in $M_{4n}(\mathbb{C})$.*
2. *There exists unitaries $v_{ij} \in M_2(\mathbb{C})$ such that $(D_{ij}v_{ij})_{i,j=1}^4$ is unitary in $M_8(\mathbb{C})$.*
3. $\delta_1 - \delta_2 - \delta_3 - \delta_4 \leq 0$.

Proof $2 \Rightarrow 1$ is trivial, and lemma 4.5 shows $1 \Rightarrow 3$. Moreover $3 \Rightarrow 2$ follows from proposition 4.13, by using that the quaternions $\mathbb{H} = \{a_1 + ia_2 + ja_3 + ka_4 \mid a_1, a_2, a_3, a_4 \in \mathbb{R}\}$ can be identified with the real subalgebra of $M_2(\mathbb{C})$ given by

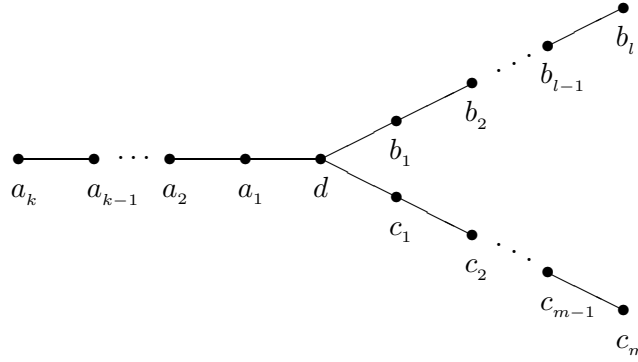
$$\left\{ \begin{pmatrix} a_1 + ia_2 & a_2 + ia_3 \\ -a_2 + ia_3 & a_1 - ia_2 \end{pmatrix} \mid a_1, a_2, a_3, a_4 \in \mathbb{R} \right\},$$

and that the matrix representation of $a \in \mathbb{H}$ is a unitary matrix if and only if $|a| = 1$, where $|a|^2 = a_1^2 + a_2^2 + a_3^2 + a_4^2$. \square

Commuting Squares Based on 3-stars

5 A Necessary and Sufficient Condition

Let Γ be the 3-star with ray length k , l , and m . We label the vertices of Γ the following way



Let Γ_0 denote the vertices of Γ . For $p, q \in \Gamma_0$, $\text{dist}(p, q)$ denotes the minimal number of edges in a path from p to q . Γ is a bi-partite graph

$$\Gamma_0 = \Gamma_{\text{even}} \cup \Gamma_{\text{odd}} \quad (\text{disjoint}),$$

where

$$\Gamma_{\text{even}} = \{p \in \Gamma_0 \mid \text{dist}(p, d) \text{ is even} \}$$

$$\Gamma_{\text{odd}} = \{p \in \Gamma_0 \mid \text{dist}(p, d) \text{ is odd} \},$$

then the adjacency matrix, Δ_Γ , of Γ is of the form

$$\Delta_\Gamma = \begin{pmatrix} 0 & G \\ G^t & 0 \end{pmatrix},$$

where the rows (resp. the columns) of G are labeled by Γ_{even} (resp. Γ_{odd}). The entries of the matrix

$$\Delta_\Gamma^2 - I = \begin{pmatrix} GG^t - I & 0 \\ 0 & G^tG - I \end{pmatrix}$$

are easily found. The off-diagonal entries are

$$(\Delta_\Gamma^2 - I)_{pq} = \begin{cases} 1 & \text{if } \text{dist}(p, q) = 2 \\ 0 & \text{otherwise} \end{cases} \quad p \neq q,$$

and the diagonal entries are

$$(\Delta_\Gamma^2 - I)_{pp} = \begin{cases} 2 & \text{if } p = d \\ 0 & \text{if } p = a_k, p = b_l \text{ or } p = c_m. \\ 1 & \text{otherwise} \end{cases}$$

In particular $(\Delta_\Gamma^2 - I)_{pq} \neq 0$ implies that $\text{dist}(p, q) = 0$ or $\text{dist}(p, q) = 2$.

We will consider commuting squares of the form

$$\begin{array}{ccc} B & \subset_{G^t G - I} & D \\ \cup_G & & \cup_G \\ A & \subset_{G G^t - I} & C \end{array} \quad (5.1)$$

Note that such a commuting square is symmetric in the sense of 1.8, because

$$G^t(GG^t - I) = (G^tG - I)G^t.$$

The rest of this section will be used to prove:

Theorem 5.1 *Let $\xi : \Gamma_0 \rightarrow \mathbb{R}_+$ be the Perron–Frobenius eigenvector for Δ_Γ , with normalization $\xi(d) = 1$. Set*

$$\alpha_1 = \xi(a_1), \quad \alpha_2 = \xi(b_1) \quad \text{and} \quad \alpha_3 = \xi(c_1), \quad (5.2)$$

and let $\lambda = \alpha_1 + \alpha_2 + \alpha_3$ be the Perron–Frobenius eigenvalue. Then Γ admits a commuting square of the form (5.1) if and only if there exists 9 vectors $(e_{ij})_{i,j=1}^3$ in \mathbb{C}^2 , for which

- (a) $\|e_{ii}\|^2 = (\lambda - \alpha_i)(\lambda\alpha_i - 1).$
- (b) $\|e_{ij}\|^2 = \alpha_i + \alpha_j - \lambda\alpha_i\alpha_j, \quad i \neq j.$
- (c) $\sum_j e_{ij} \otimes \bar{e}_{ij} = \alpha_i I_2, \quad i = 1, 2, 3.$
- (d) $\sum_i e_{ij} \otimes \bar{e}_{ij} = \alpha_j I_2, \quad j = 1, 2, 3.$

where I_2 is the unit 2×2 -matrix.

Proof of the necessity of (a), (b), (c) and (d):

First we consider the case $k, l, m \geq 2$. Using the eigenvector equation $\Delta_\Gamma \xi = \lambda \xi$ at the vertices a_1 , b_1 and c_1 one gets

$$\xi(a_2) = \lambda\alpha_1 - 1, \quad \xi(b_2) = \lambda\alpha_2 - 1, \quad \xi(c_2) = \lambda\alpha_3 - 1. \quad (5.3)$$

Particularly $\lambda\alpha_i - 1 > 0$, $i = 1, 2, 3$.

Assume that there is a commuting square of the form (5.1). The sets of minimal central projections $c(A)$ and $c(C)$ of A and C are labeled by the elements of Γ_{even} , and the sets of minimal central

projections $c(B)$ and $c(D)$ of B and D are labeled by the elements of Γ_{odd} . The bi-unitary condition (1.9) gives the existence of

$$u = \bigoplus_{(p,s)} u^{(p,s)}, \quad v = \bigoplus_{(q,r)} v^{(q,r)},$$

where $u^{(p,s)}$ and $v^{(q,r)}$ are block matrices

$$u^{(p,s)} = \left(u_{qr}^{(p,s)} \right)_{q,r} \quad \text{and} \quad v^{(q,r)} = \left(v_{ps}^{(q,r)} \right)_{p,s}.$$

The only sets of indices (p, q, r, s) which occur are those, that can be completed to a cycle of length 4, $p - r - s - q - p$, via the given inclusion pattern. Moreover each block $u_{qr}^{(p,s)}$ is a scalar unless $q = r = d$, in which case $u_{qr}^{(p,s)}$ is a 1×2 -matrix, because the edge dd is the only multiple edge coming from $GG^t - I$ or $G^tG - I$, and the multiplicity is 2. Finally

$$v_{ps}^{(q,r)} = w(p, q, r, s) \left(u_{qr}^{(p,s)} \right)^t, \quad (5.4)$$

where

$$w(p, q, r, s) = \sqrt{\frac{\xi(p)\xi(s)}{\xi(q)\xi(r)}}. \quad (5.5)$$

The possible 4-cycles $p - r - s - q - p$ are determined by the two “vertical” edges pq and rs from Γ , but not all pairs (pr, qs) will occur. We concentrate on the 6 edges

$$da_1, \quad db_1, \quad dc_1, \quad a_2a_1, \quad b_2b_1, \quad c_2c_1,$$

in Γ , which connect vertices at a distance of at most 2 from the central vertex d .

Figure 1 shows which combinations (pr, qs) occur, and the number of dots indicates the size of the corresponding block $u_{qr}^{(p,s)}$.

$rs \backslash pq$	da_1	db_1	dc_1	a_2a_1	b_2b_1	c_2c_1
da_1	• •	• •	• •	•	•	•
db_1	• •	• •	• •	•	•	•
dc_1	• •	• •	• •	•	•	•
a_2a_1	•	•	•	?		
b_2b_1	•	•	•		?	
c_2c_1	•	•	•			?

Figure 1. Blocks of u indexed by edges close to d

The 3 question marks each represent a dot if $k, l, m \geq 3$, but if the a -ray, the b -ray or the c -ray has length 2, the corresponding question mark represents an empty box, i.e. the pair (pq, rs) does not correspond to a 4-cycle.

The corresponding entries, $u_{qr}^{(p,s)}$, of u are given by

$$f_{ij} \in \mathbb{C}^2 \text{ (row vectors) and } \sigma_{ij}, \tau_{ij} \in \mathbb{C}$$

as in figure 2.

$rs \backslash pq$	da_1	db_1	dc_1	a_2a_1	b_2b_1	c_2c_1
da_1	f_{11}	f_{12}	f_{13}	σ_{11}	σ_{12}	σ_{13}
db_1	f_{21}	f_{22}	f_{23}	σ_{21}	σ_{22}	σ_{23}
dc_1	f_{31}	f_{32}	f_{33}	σ_{31}	σ_{32}	σ_{33}
a_2a_1	τ_{11}	τ_{12}	τ_{13}	?		
b_2b_1	τ_{21}	τ_{22}	τ_{23}		?	
c_2c_1	τ_{31}	τ_{32}	τ_{33}			?

Figure 2. Entries of u

Let $f'_{ij} \in \mathbb{C}^2$ (column vectors) and $\sigma'_{ij}, \tau'_{ij} \in \mathbb{C}$ be the corresponding blocks, $v_{p,s}^{(qr)}$, of v . By the transformation formulas (5.4) and (5.5), together with $\xi(d) = 1$ and (5.2), (5.3) we get

$$f'_{ij} = \sqrt{\frac{\alpha_i}{\alpha_j}} f_{ij}^t \quad (5.6)$$

$$\sigma'_{ij} = \sqrt{\frac{\alpha_j}{\alpha_i(\lambda\alpha_j - 1)}} \sigma_{ij} \quad (5.7)$$

$$\tau'_{ij} = \sqrt{\frac{\alpha_j(\lambda\alpha_i - 1)}{\alpha_i}} \tau_{ij} \quad (5.8)$$

The unitary summands $u^{(p,s)}$ of u for $(p, s) = (d, a_1)$ (resp. (d, b_1) and (d, c_1)) are

$$u^{(d,a_1)} = \begin{pmatrix} f_{11} & \sigma_{11} \\ f_{21} & \sigma_{21} \\ f_{31} & \sigma_{31} \end{pmatrix}, \quad u^{(d,b_1)} = \begin{pmatrix} f_{12} & \sigma_{12} \\ f_{22} & \sigma_{22} \\ f_{32} & \sigma_{32} \end{pmatrix}, \quad u^{(d,c_1)} = \begin{pmatrix} f_{13} & \sigma_{13} \\ f_{23} & \sigma_{23} \\ f_{33} & \sigma_{33} \end{pmatrix}. \quad (5.9)$$

Similarly the unitary summands, $v^{(q,r)}$, of v for $(q, r) = (a_1, d)$ (resp. (b_1, d) and (c_1, d)) are

$$\begin{aligned}
v^{(a_1, d)} &= \begin{pmatrix} f'_{11} & f'_{12} & f'_{13} \\ \tau'_{11} & \tau'_{12} & \tau'_{13} \end{pmatrix} \\
v^{(b_1, d)} &= \begin{pmatrix} f'_{21} & f'_{22} & f'_{23} \\ \tau'_{21} & \tau'_{22} & \tau'_{23} \end{pmatrix} \\
v^{(c_1, d)} &= \begin{pmatrix} f'_{31} & f'_{32} & f'_{33} \\ \tau'_{31} & \tau'_{32} & \tau'_{33} \end{pmatrix}.
\end{aligned} \tag{5.10}$$

Since $\text{dist}(a_2, b_1) = 3$, there is only one pair (q, r) , such that $a_2 - r - b_1 - q - a_2$ is a 4-cycle with the given inclusion matrices, namely $r = d$ and $q = a_1$. Hence

$$u^{(a_2, b_1)} = u^{(a_2, b_1)}_{a_1 d} = \tau_{12}$$

is a 1×1 -summand of u . The same argument shows that τ_{ij} is a 1×1 unitary of u whenever $i \neq j$, and σ'_{ij} is a 1×1 unitary of v when $i \neq j$. In particular

$$|\sigma'_{ij}| = |\tau_{ij}| = 1, \quad i \neq j. \tag{5.11}$$

Hence by (5.7) and (5.8)

$$\begin{aligned}
|\sigma_{ij}| &= \sqrt{\frac{\alpha_i(\lambda\alpha_j - 1)}{\alpha_j}}, \quad i \neq j \\
|\tau'_{ij}| &= \sqrt{\frac{\alpha_j(\lambda\alpha_i - 1)}{\alpha_i}}, \quad i \neq j.
\end{aligned}$$

Using that the 3 matrices in (5.9) are unitary, one has

$$\|f_{ij}\|^2 = 1 - |\sigma_{ij}|^2 = \frac{1}{\alpha_j}(\alpha_i + \alpha_j - \lambda\alpha_i\alpha_j), \quad i \neq j,$$

and

$$\begin{aligned}
\|f_{ii}\|^2 &= 1 - |\sigma_{ii}|^2 \\
&= |\sigma_{ji}|^2 + |\sigma_{ki}|^2 \quad i \neq j \neq k \neq i \\
&= \frac{1}{\alpha_i}(\alpha_j + \alpha_k)(\lambda\alpha_i - 1) \\
&= \frac{1}{\alpha_i}(\lambda - \alpha_i)(\lambda\alpha_i - 1).
\end{aligned}$$

Set $e_{ij} = \sqrt{\alpha_j}f_{ij} \in \mathbb{C}^2$. Then

$$\|e_{ij}\|^2 = \begin{cases} (\lambda - \alpha_i)(\lambda\alpha_i - 1), & i = j \\ \alpha_i + \alpha_j - \lambda\alpha_i\alpha_j, & i \neq j \end{cases}$$

Moreover, using that the rows of a unitary matrix are orthogonal vectors, one gets from (5.9)

$$\sum_i e_{ij} \otimes \bar{e}_{ij} = \alpha_j \sum_i f_{ij} \otimes \bar{f}_{ij} = \alpha_j \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad j = 1, 2, 3.$$

Furthermore, since

$$e_{ij} = \frac{1}{\sqrt{\alpha_i}} \left(f'_{ij} \right)^t,$$

and since the two first rows of each of the matrices (5.10) are orthonormal, we have

$$\sum_j e_{ij} \otimes \bar{e}_{ij} = \alpha_i \sum_j \left(f'_{ij}\right)^t \otimes \left(\bar{f}'_{ij}\right)^t = \alpha_i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = 1, 2, 3.$$

This proves the necessity of (a), (b), (c) and (d) in the case $k, l, m \geq 2$.

If one of the rays, say the c -ray, has length 1, but $k, l \geq 2$, then

$$\alpha_3 = \frac{\xi(d)}{\lambda} = \frac{1}{\lambda}, \quad \text{i.e. } \lambda\alpha_3 - 1 = 0.$$

Figure 2 reduces to

$\begin{array}{c} rs \\ pq \end{array}$	da_1	db_1	dc_1	a_2a_1	b_2b_1
da_1	f_{11}	f_{12}	f_{13}	σ_{11}	σ_{12}
db_1	f_{21}	f_{22}	f_{23}	σ_{21}	σ_{22}
dc_1	f_{31}	f_{32}		σ_{31}	σ_{32}
a_2a_1	τ_{11}	τ_{12}	τ_{13}	?	
b_2b_1	τ_{21}	τ_{22}	τ_{23}		?

Figure 3.

Thus $u^{(d,c_1)}$ and $v^{(c_1,d)}$ in (5.9) and (5.10) reduce to 2×2 matrices

$$u^{(d,c_1)} = \begin{pmatrix} f_{13} \\ f_{23} \end{pmatrix}, \quad v^{(c_1,d)} = (f'_{31}, f'_{32}).$$

However this case involves the same computations as in the case $k, l, m \geq 2$, if we set $f_{33} = 0$, $\sigma_{13} = \sigma_{23} = 0$ and $\tau_{31} = \tau_{32} = 0$, because $\lambda\alpha_3 - 1 = 0$. This proves the necessity part of theorem 5.1.

Proof of sufficiency of (a), (b), (c) and (d):

Assume that $(e_{ij})_{i,j=1,2,3}$ are 9 vectors in \mathbb{C}^2 satisfying (a), (b), (c) and (d). Since any orthonormal set in \mathbb{C}^3 can be completed to an orthonormal basis, (d) implies that there exists $\sigma_{ij} \in \mathbb{C}$, such that

$$u_i = \begin{pmatrix} \frac{1}{\sqrt{\alpha_i}} e_{1i} & \sigma_{1i} \\ \frac{1}{\sqrt{\alpha_i}} e_{2i} & \sigma_{2i} \\ \frac{1}{\sqrt{\alpha_i}} e_{3i} & \sigma_{3i} \end{pmatrix} \quad i = 1, 2, 3, \quad (5.12)$$

are 3 unitary 3×3 matrices, and by (c) there exists $\rho_{ij} \in \mathbb{C}$, such that

$$v_i = \begin{pmatrix} \frac{1}{\sqrt{\alpha_i}} e_{i1}^t & \frac{1}{\sqrt{\alpha_i}} e_{i2}^t & \frac{1}{\sqrt{\alpha_i}} e_{i3}^t \\ \rho_{i1} & \rho_{i2} & \rho_{i3} \end{pmatrix} \quad i = 1, 2, 3, \quad (5.13)$$

are 3 unitary 3×3 matrices. By multiplying the last column in each u_i and the last row in each v_i by suitable scalars of modulus 1, we can obtain

$$\sigma_{ii} \geq 0 \quad \text{and} \quad \rho_{ii} \geq 0, \quad i = 1, 2, 3.$$

From (a) and (b) and the unitarity of u_i and v_i we get

$$|\sigma_{ij}|^2 = 1 - \frac{1}{\alpha_j} \|e_{ij}\|^2 = \begin{cases} \frac{\lambda}{\alpha_j} (\alpha_j^2 - \lambda\alpha_j + 1), & i = j \\ \frac{\alpha_i}{\alpha_j} (\lambda\alpha_j - 1), & i \neq j \end{cases} \quad (5.14)$$

and

$$|\rho_{ij}|^2 = 1 - \frac{1}{\alpha_i} \|e_{ij}\|^2 = \begin{cases} \frac{\lambda}{\alpha_i} (\alpha_i^2 - \lambda\alpha_i + 1), & i = j \\ \frac{\alpha_j}{\alpha_i} (\lambda\alpha_i - 1), & i \neq j. \end{cases} \quad (5.15)$$

Let δ_1, δ_2 and δ_3 be the coordinates of the Perron–Frobenius vector, ξ , at the endpoints of the rays,

$$\delta_1 = \xi(a_k), \quad \delta_2 = \xi(b_l), \quad \delta_3 = \xi(c_m).$$

By chapter I (2.8), $\delta_i = \sqrt{\alpha_i^2 - \lambda\alpha_i + 1}$, $i = 1, 2, 3$, so for $i = j$ the above formulas reduce to

$$|\sigma_{ii}|^2 = |\rho_{ii}|^2 = \frac{\lambda\delta_i^2}{\alpha_i}, \quad i = 1, 2, 3,$$

and since $\sigma_{ii} \geq 0$ and $\rho_{ii} \geq 0$ we have

$$\sigma_{ii} = \rho_{ii} = \sqrt{\frac{\lambda\delta_i^2}{\alpha_i}}, \quad i = 1, 2, 3. \quad (5.16)$$

By (5.14) and (5.15) there exists scalars σ'_{ij} and ρ'_{ij} with $|\sigma'_{ij}| = |\rho'_{ij}| = 1$, $i \neq j$, such that

$$\sigma_{ij} = \sqrt{\frac{\alpha_i}{\alpha_j} (\lambda\alpha_j - 1)} \sigma'_{ij}, \quad i \neq j, \quad (5.17)$$

$$\rho_{ij} = \sqrt{\frac{\alpha_j}{\alpha_i} (\lambda\alpha_i - 1)} \rho'_{ij}, \quad i \neq j. \quad (5.18)$$

Let $R_n(\lambda)$ be the polynomials defined by

$$R_0(\lambda) = 1, \quad R_1(\lambda) = \lambda, \quad R_{n+1}(\lambda) = \lambda R_n(\lambda) - R_{n-1}(\lambda), \quad n \geq 1,$$

as in chapter I 2.1. Then by (2.4) the Perron–Frobenius vector, ξ , on Γ is given by

$$\begin{aligned} \xi(d) &= 1 \\ \xi(a_j) &= \frac{R_{k-j}(\lambda)}{R_k(\lambda)}, \quad j = 1, \dots, k \\ \xi(b_j) &= \frac{R_{l-j}(\lambda)}{R_l(\lambda)}, \quad j = 1, \dots, l \\ \xi(c_j) &= \frac{R_{m-j}(\lambda)}{R_m(\lambda)}, \quad j = 1, \dots, m. \end{aligned}$$

Set

$$\alpha_{1j} = \xi(a_j), \quad \alpha_{2j} = \xi(b_j), \quad \alpha_{3j} = \xi(c_j).$$

In particular

$$\begin{aligned} \alpha_1 &= \alpha_{11}, & \alpha_2 &= \alpha_{21}, & \alpha_3 &= \alpha_{31}, \\ \delta_1 &= \alpha_{1k}, & \delta_2 &= \alpha_{2l}, & \delta_3 &= \alpha_{3m}. \end{aligned}$$

We proceed to construct $u = \bigoplus u^{(p,s)}$ and $v = \bigoplus v^{(q,r)}$ satisfying the bi-unitary condition, i.e. u and v are unitaries and

$$v_{ps}^{(q,r)} = \sqrt{\frac{\xi(p)\xi(s)}{\xi(q)\xi(r)}} u_{qr}^{(p,s)}$$

for all possible 4-cycles $p - r - s - q - p$. Since the matrix elements

$$(GG^t - I)_{p,r}, \quad p, r \in \Gamma_{\text{even}}$$

and

$$(G^tG - I)_{q,s}, \quad q, s \in \Gamma_{\text{odd}}$$

vanish unless $\text{dist}(p, r) \leq 2$ and $\text{dist}(q, s) \leq 2$, the possible pairs of edges (pr, qs) which define 4-cycles must either be on the same ray of Γ , or they must connect vertices of Γ with distance at most 2 from the central vertex d . In the latter case we define the entries of u and v by figure 4 and figure 5 below. For the moment we will assume that $k, l, m \geq 3$.

$\begin{array}{c} rs \\ pq \end{array}$	da_1	db_1	dc_1	a_2a_1	b_2b_1	c_2c_1
da_1	$\frac{1}{\sqrt{\alpha_1}}e_{11}$	$\frac{1}{\sqrt{\alpha_2}}e_{12}$	$\frac{1}{\sqrt{\alpha_3}}e_{13}$	$\sqrt{\frac{\lambda\delta_1^2}{\alpha_1}}$	σ_{12}	σ_{13}
db_1	$\frac{1}{\sqrt{\alpha_1}}e_{21}$	$\frac{1}{\sqrt{\alpha_2}}e_{22}$	$\frac{1}{\sqrt{\alpha_3}}e_{23}$	σ_{21}	$\sqrt{\frac{\lambda\delta_2^2}{\alpha_2}}$	σ_{23}
dc_1	$\frac{1}{\sqrt{\alpha_1}}e_{31}$	$\frac{1}{\sqrt{\alpha_2}}e_{32}$	$\frac{1}{\sqrt{\alpha_3}}e_{33}$	σ_{31}	σ_{32}	$\sqrt{\frac{\lambda\delta_3^2}{\alpha_3}}$
a_2a_1	$\sqrt{\frac{\lambda\delta_1^2}{\alpha_1\alpha_{12}}}$	ρ'_{12}	ρ'_{13}	$-\sqrt{\frac{\alpha_{13}}{\alpha_1\alpha_{12}}}$		
b_2b_1	ρ'_{21}	$\sqrt{\frac{\lambda\delta_2^2}{\alpha_2\alpha_{22}}}$	ρ'_{23}		$-\sqrt{\frac{\alpha_{23}}{\alpha_2\alpha_{22}}}$	
c_2c_1	ρ'_{31}	ρ'_{32}	$\sqrt{\frac{\lambda\delta_3^2}{\alpha_3\alpha_{32}}}$			$-\sqrt{\frac{\alpha_{33}}{\alpha_3\alpha_{32}}}$

Figure 4. Entries of u near the central vertex.

$\begin{array}{c} rs \\ pq \end{array}$	da_1	db_1	dc_1	a_2a_1	b_2b_1	c_2c_1
da_1	$\frac{1}{\sqrt{\alpha_1}}e_{11}^t$	$\frac{1}{\sqrt{\alpha_1}}e_{12}^t$	$\frac{1}{\sqrt{\alpha_1}}e_{13}^t$	$\sqrt{\frac{\lambda\delta_1^2}{\alpha_1\alpha_{12}}}$	σ'_{12}	σ'_{13}
db_1	$\frac{1}{\sqrt{\alpha_2}}e_{21}^t$	$\frac{1}{\sqrt{\alpha_2}}e_{22}^t$	$\frac{1}{\sqrt{\alpha_2}}e_{23}^t$	σ'_{21}	$\sqrt{\frac{\lambda\delta_2^2}{\alpha_2\alpha_{22}}}$	σ'_{23}
dc_1	$\frac{1}{\sqrt{\alpha_3}}e_{31}^t$	$\frac{1}{\sqrt{\alpha_3}}e_{32}^t$	$\frac{1}{\sqrt{\alpha_3}}e_{33}^t$	σ'_{31}	σ'_{32}	$\sqrt{\frac{\lambda\delta_3^2}{\alpha_3\alpha_{32}}}$
a_2a_1	$\sqrt{\frac{\lambda\delta_1^2}{\alpha_1}}$	ρ_{12}	ρ_{13}	$-\sqrt{\frac{\alpha_{13}}{\alpha_1\alpha_{12}}}$		
b_2b_1	ρ_{21}	$\sqrt{\frac{\lambda\delta_2^2}{\alpha_2}}$	ρ_{23}		$-\sqrt{\frac{\alpha_{23}}{\alpha_2\alpha_{22}}}$	
c_2c_1	ρ_{31}	ρ_{32}	$\sqrt{\frac{\lambda\delta_3^2}{\alpha_3}}$			$-\sqrt{\frac{\alpha_{33}}{\alpha_3\alpha_{32}}}$

Figure 5. Entries of v near the central vertex.

The empty entries in u resp. v correspond to pairs (pq, rs) which cannot be completed to a 4-cycle.

The entries of u and v for pairs of edges (pq, rs) on the a -ray, are defined by the figures 6, 7, 8 and 9 below

$rs \backslash pq$	da_1	$a_2 a_1$	$a_2 a_3$	$a_4 a_3$	$a_4 a_5$	$a_6 a_5$
da_1	$\frac{1}{\sqrt{\alpha_1}} e_{11}$	$\sqrt{\frac{\lambda \delta_1^2}{\alpha_1}}$	1			
$a_2 a_1$	$\sqrt{\frac{\lambda \delta_1^2}{\alpha_1 \alpha_{12}}}$	$-\sqrt{\frac{\alpha_{13}}{\alpha_1 \alpha_{12}}}$	$\sqrt{\frac{\lambda \delta_1^2}{\alpha_{12} \alpha_{13}}}$	$\sqrt{\frac{\alpha_{11} \alpha_{14}}{\alpha_{12} \alpha_{13}}}$		
$a_2 a_3$	$\sqrt{\frac{\alpha_{13}}{\alpha_1 \alpha_{12}}}$	$\sqrt{\frac{\lambda \delta_1^2}{\alpha_1 \alpha_{12}}}$	$-\sqrt{\frac{\alpha_{11} \alpha_{14}}{\alpha_{12} \alpha_{13}}}$	$\sqrt{\frac{\lambda \delta_1^2}{\alpha_{12} \alpha_{13}}}$	1	
$a_4 a_3$		1	$\sqrt{\frac{\lambda \delta_1^2}{\alpha_{13} \alpha_{14}}}$	$-\sqrt{\frac{\alpha_{12} \alpha_{15}}{\alpha_{13} \alpha_{14}}}$	$\sqrt{\frac{\lambda \delta_1^2}{\alpha_{14} \alpha_{15}}}$	$\sqrt{\frac{\alpha_{13} \alpha_{16}}{\alpha_{14} \alpha_{15}}}$
$a_4 a_5$			$\sqrt{\frac{\alpha_{12} \alpha_{15}}{\alpha_{13} \alpha_{14}}}$	$\sqrt{\frac{\lambda \delta_1^2}{\alpha_{13} \alpha_{14}}}$	$-\sqrt{\frac{\alpha_{13} \alpha_{16}}{\alpha_{14} \alpha_{15}}}$	$\sqrt{\frac{\lambda \delta_1^2}{\alpha_{14} \alpha_{15}}}$
$a_6 a_5$				1	$\sqrt{\frac{\lambda \delta_1^2}{\alpha_{15} \alpha_{16}}}$	$-\sqrt{\frac{\alpha_{14} \alpha_{17}}{\alpha_{15} \alpha_{16}}}$

Figure 6. Entries of u ; a -ray near the central vertex

$rs \backslash pq$	da_1	$a_2 a_1$	$a_2 a_3$	$a_4 a_3$	$a_4 a_5$	$a_6 a_5$
da_1	$\frac{1}{\sqrt{\alpha_1}} e_{11}^t$	$\sqrt{\frac{\lambda \delta_1^2}{\alpha_1 \alpha_{12}}}$	$\sqrt{\frac{\alpha_{13}}{\alpha_1 \alpha_{12}}}$			
$a_2 a_1$	$\sqrt{\frac{\lambda \delta_1^2}{\alpha_1}}$	$-\sqrt{\frac{\alpha_{13}}{\alpha_1 \alpha_{12}}}$	$\sqrt{\frac{\lambda \delta_1^2}{\alpha_1 \alpha_{12}}}$	1		
$a_2 a_3$	1	$\sqrt{\frac{\lambda \delta_1^2}{\alpha_{12} \alpha_{13}}}$	$-\sqrt{\frac{\alpha_{11} \alpha_{14}}{\alpha_{12} \alpha_{13}}}$	$\sqrt{\frac{\lambda \delta_1^2}{\alpha_{13} \alpha_{14}}}$	$\sqrt{\frac{\alpha_{12} \alpha_{15}}{\alpha_{13} \alpha_{14}}}$	
$a_4 a_3$		$\sqrt{\frac{\alpha_{11} \alpha_{14}}{\alpha_{12} \alpha_{13}}}$	$\sqrt{\frac{\lambda \delta_1^2}{\alpha_{12} \alpha_{13}}}$	$-\sqrt{\frac{\alpha_{12} \alpha_{15}}{\alpha_{13} \alpha_{14}}}$	$\sqrt{\frac{\lambda \delta_1^2}{\alpha_{13} \alpha_{14}}}$	1
$a_4 a_5$			1	$\sqrt{\frac{\lambda \delta_1^2}{\alpha_{14} \alpha_{15}}}$	$-\sqrt{\frac{\alpha_{13} \alpha_{16}}{\alpha_{14} \alpha_{15}}}$	$\sqrt{\frac{\lambda \delta_1^2}{\alpha_{15} \alpha_{16}}}$
$a_6 a_5$				$\sqrt{\frac{\alpha_{13} \alpha_{16}}{\alpha_{14} \alpha_{15}}}$	$\sqrt{\frac{\lambda \delta_1^2}{\alpha_{14} \alpha_{15}}}$	$-\sqrt{\frac{\alpha_{14} \alpha_{17}}{\alpha_{15} \alpha_{16}}}$

Figure 7. Entries of v ; a -ray near the central vertex

$rs \backslash pq$	$a_{k-4}a_{k-5}$	$a_{k-4}a_{k-3}$	$a_{k-2}a_{k-3}$	$a_{k-2}a_{k-1}$	$a_k a_{k-1}$
$a_{k-4}a_{k-5}$	$-\sqrt{\frac{\delta_{16}\delta_{13}}{\delta_{15}\delta_{14}}}$	$\sqrt{\frac{\lambda\delta_1^2}{\delta_{14}\delta_{13}}}$	$\sqrt{\frac{\delta_{15}\delta_{12}}{\delta_{14}\delta_{13}}}$		
$a_{k-4}a_{k-3}$	$\sqrt{\frac{\lambda\delta_1^2}{\delta_{15}\delta_{14}}}$	$-\sqrt{\frac{\delta_{15}\delta_{12}}{\delta_{14}\delta_{13}}}$	$\sqrt{\frac{\lambda\delta_1^2}{\delta_{14}\delta_{13}}}$	1	
$a_{k-2}a_{k-3}$	1	$\sqrt{\frac{\lambda\delta_1^2}{\delta_{13}\delta_{12}}}$	$-\sqrt{\frac{\delta_{14}\delta_{11}}{\delta_{13}\delta_{12}}}$	$\sqrt{\frac{\lambda\delta_1^2}{\delta_{12}\delta_{11}}}$	$\sqrt{\frac{\delta_{13}\delta_{10}}{\delta_{12}\delta_{11}}}$
$a_{k-2}a_{k-1}$		$\sqrt{\frac{\delta_{14}\delta_{11}}{\delta_{13}\delta_{12}}}$	$\sqrt{\frac{\lambda\delta_1^2}{\delta_{13}\delta_{12}}}$	$-\sqrt{\frac{\delta_{13}\delta_{10}}{\delta_{12}\delta_{11}}}$	$\sqrt{\frac{\lambda\delta_1^2}{\delta_{12}\delta_{11}}}$
$a_k a_{k-1}$			1	1	

Figure 8.

Entries of u : a -ray near the end vertex for k even, $\delta_{1i} = \alpha_{1,k-i}$, $i = 0, 1, \dots, k-1$. The corresponding entries of v are given by the mirror image of figure 8 in the main diagonal.

$\begin{array}{c} rs \\ pq \end{array}$	$a_{k-5}a_{k-4}$	$a_{k-3}a_{k-4}$	$a_{k-3}a_{k-2}$	$a_{k-1}a_{k-2}$	$a_{k-1}a_k$
$a_{k-5}a_{k-4}$	$-\sqrt{\frac{\delta_{16}\delta_{13}}{\delta_{15}\delta_{14}}}$	$\sqrt{\frac{\lambda\delta_1^2}{\delta_{15}\delta_{14}}}$	1		
$a_{k-3}a_{k-4}$	$\sqrt{\frac{\lambda\delta_1^2}{\delta_{14}\delta_{13}}}$	$-\sqrt{\frac{\delta_{15}\delta_{12}}{\delta_{14}\delta_{13}}}$	$\sqrt{\frac{\lambda\delta_1^2}{\delta_{13}\delta_{12}}}$	$\sqrt{\frac{\delta_{14}\delta_{11}}{\delta_{13}\delta_{12}}}$	
$a_{k-3}a_{k-2}$	$\sqrt{\frac{\delta_{15}\delta_{12}}{\delta_{14}\delta_{13}}}$	$\sqrt{\frac{\lambda\delta_1^2}{\delta_{14}\delta_{13}}}$	$-\sqrt{\frac{\delta_{14}\delta_{11}}{\delta_{13}\delta_{12}}}$	$\sqrt{\frac{\lambda\delta_1^2}{\delta_{13}\delta_{12}}}$	1
$a_{k-1}a_{k-2}$		1	$\sqrt{\frac{\lambda\delta_1^2}{\delta_{12}\delta_{11}}}$	$-\sqrt{\frac{\delta_{13}\delta_{10}}{\delta_{12}\delta_{11}}}$	1
$a_{k-1}a_k$			$\sqrt{\frac{\delta_{13}\delta_{10}}{\delta_{12}\delta_{11}}}$	$\sqrt{\frac{\lambda\delta_1^2}{\delta_{12}\delta_{11}}}$	

Figure 9.

Entries of u : a -ray near the end vertex for k odd, $\delta_{1i} = \alpha_{1,k-i}$, $i = 0, 1, \dots, k-1$. The corresponding entries of v are given by the mirror image of figure 9 in the main diagonal.

The entries of u and v , for pairs of edges (pq, rs) on the b -ray or on the c -ray, are defined the same way, with trivial changes of the indices.

It is easy to check that the entries of u and v satisfy the transformation formula (5.4). The direct summands $u^{(p,s)}$ of u are marked by the framing in fig. 6, fig. 8 and fig. 9 for (p, s) both on the a -ray (and similarly for the other rays). These are either 1×1 - or 2×2 -matrices. To see that the 2×2 -matrices are unitary, we have to check

$$\lambda\delta_1^2 = \alpha_{1,j+1}\alpha_{1,j+2} - \alpha_{1,j}\alpha_{1,j+3}, \quad j = 1, 2, \dots, k-3. \quad (5.19)$$

From the proof of lemma 3.1 in chapter I we have the identity

$$R_{m+1}(\lambda)^2 - R_{m+2}(\lambda)R_m(\lambda) = 1, \quad m = 0, 1, 2, \dots$$

Using the recursion formula for R_m :

$$R_{m+2}(\lambda)R_{m+1}(\lambda) - R_{m+3}(\lambda)R_m(\lambda) = (\lambda R_{m+1}(\lambda) - R_m(\lambda))R_{m+1}(\lambda) - (\lambda R_{m+2}(\lambda) - R_{m+1}(\lambda))R_m(\lambda).$$

Hence

$$R_{m+2}(\lambda)R_{m+1}(\lambda) - R_{m+3}(\lambda)R_m(\lambda) = \lambda, \quad m = 0, 1, 2, \dots$$

If we use

$$\delta_1 = \frac{1}{R_k(\lambda)} \quad \text{and} \quad \alpha_{1,j} = \frac{R_{k-j}(\lambda)}{R_k(\lambda)},$$

(5.19) follows by putting $m = k - j - 3$.

The only summands, $u^{(p,s)}$, of u , which are not visible in fig. 6, fig. 8, fig. 9 or the corresponding figures for the b - and the c -rays, are three 3×3 -matrices

$$u^{(d,a_1)}, \quad u^{(d,b_1)}, \quad u^{(d,c_1)}$$

and six 1×1 -matrices

$$u^{(a_2,b_1)}, \quad u^{(a_2,c_1)}, \quad u^{(b_2,a_1)}, \quad u^{(b_2,c_1)}, \quad u^{(c_2,a_1)}, \quad u^{(c_2,b_1)}.$$

By the previous analysis, the 3×3 -matrices are the unitaries u_i , $i = 1, 2, 3$ in (5.12), and the 1×1 -matrices are given by the scalars $(\rho_{ij})_{i \neq j}$ of modulus 1. Hence $u = \bigoplus u^{(p,s)}$ is unitary.

Similarly one gets, that the only summands, $v^{(q,r)}$, of v , which are not visible in fig. 6 and the mirror images of fig. 8 and fig. 9 (or the corresponding diagrams for the b - and c -ray) are the three 3×3 unitaries v_i , $i = 1, 2, 3$ in (5.13) and six 1×1 -matrices, given by the scalars $(\sigma'_{ij})_{i \neq j}$ of modulus 1, so also $v = \bigoplus v^{(q,r)}$ is unitary.

This proves that (a), (b), (c) and (d) of theorem 5.1 implies the existence of a commuting square with the inclusions given by (5.1), for $k, l, m \geq 3$.

The cases where one or more of the numbers k, l, m are less than 3 follows in the same way, by appropriate cancellations in fig. 4–9.

If $k = 2$, fig. 6 and fig. 7 degenerate to

$rs \backslash pq$	da_1	$a_2 a_1$		$rs \backslash pq$	da_1	$a_2 a_1$
$a_2 a_1$	$\frac{1}{\sqrt{\alpha_1}} e_{11}$	$\sqrt{\frac{\lambda \delta_1^2}{\alpha_1}}$	and	$a_2 a_1$	$\frac{1}{\sqrt{\alpha_1}} e_{11}$	1
da_1	1			da_1	$\sqrt{\frac{\lambda \delta_1^2}{\alpha_1}}$	

and for $k = 1$ all the elements of fig. 6 and fig. 7 disappear. The cases, where k, l, m are not all greater than or equal to 3, also impose cancellations in fig. 4 and fig. 5, as described in the proof of the necessity of (a), (b), (c) and (d), but it is not hard to check, that the bi-unitary condition, for the pair (u, v) is also satisfied in these cases. This completes the proof of theorem 5.1.

6 Solution to the Vector Problem

In this section we will prove

Theorem 6.1

(1) Let $\alpha_1, \alpha_2, \alpha_3 > 0$ and put $\lambda = \alpha_1 + \alpha_2 + \alpha_3$. Then there exists 9 vectors, $(e_{ij})_{i,j=1}^3$ in a two dimensional complex Hilbert space, \mathcal{H} , such that

$$\begin{aligned} (a) \quad \|e_{ii}\|^2 &= (\lambda - \alpha_i)(\lambda\alpha_i - 1) \\ (b) \quad \|e_{ij}\|^2 &= \alpha_i + \alpha_j - \lambda\alpha_i\alpha_j, \quad i \neq j \\ (c) \quad \sum_{j=1}^3 e_{ij} \otimes \bar{e}_{ij} &= \alpha_i I_{\mathcal{H}}, \quad i = 1, 2, 3 \\ (d) \quad \sum_{i=1}^3 e_{ij} \otimes \bar{e}_{ij} &= \alpha_j I_{\mathcal{H}}, \quad j = 1, 2, 3 \end{aligned}$$

if and only if

(2)

$$\begin{aligned} (i) \quad \lambda\alpha_i - 1 &\geq 0, \quad i = 1, 2, 3 \\ (ii) \quad \alpha_i^2 - \lambda\alpha_i + 1 &\geq 0, \quad i = 1, 2, 3 \\ (iii) \quad 4\lambda\alpha_1\alpha_2\alpha_3 - 4(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) + 3 &\geq 0 \end{aligned}$$

Remark 6.2 Condition (i) is clearly a necessary condition, because $\|e_{ii}\|^2 = (\lambda - \alpha_i)(\lambda\alpha_i - 1)$ and $\lambda > \alpha_i$.

Condition (ii) is also necessary, since (b) implies that $e_{ii} \otimes \bar{e}_{ii} \leq \alpha_i I_{\mathcal{H}}$. Therefore $\|e_{ii}\|^2 \leq \alpha_i$. By (a)

$$\alpha_i - \|e_{ii}\|^2 = \lambda(\alpha_i^2 - \lambda\alpha_i + 1).$$

This shows (ii).

The most difficult part of the proof, is to show that (iii) is also a necessary condition.

Reformulation of condition (iii) in Theorem 6.1

Set

$$\gamma_i = \alpha_i^2 - \lambda\alpha_i + 1, \quad i = 1, 2, 3$$

Note that

$$\begin{aligned} \gamma_1 &= 1 - \alpha_1\alpha_2 - \alpha_1\alpha_3 \\ \gamma_2 &= 1 - \alpha_1\alpha_2 - \alpha_2\alpha_3 \\ \gamma_3 &= 1 - \alpha_1\alpha_3 - \alpha_2\alpha_3. \end{aligned}$$

We compute

$$\begin{aligned}
& 2\gamma_1\gamma_2 + 2\gamma_1\gamma_3 + 2\gamma_2\gamma_3 - \gamma_1^2 - \gamma_2^2 - \gamma_3^2 \\
&= \gamma_1(\gamma_2 + \gamma_3 - \gamma_1) + \gamma_2(\gamma_1 + \gamma_3 - \gamma_2) + \gamma_3(\gamma_1 + \gamma_2 - \gamma_3) \\
&= (1 - \alpha_1\alpha_2 - \alpha_1\alpha_3)(1 - 2\alpha_2\alpha_3) + (1 - \alpha_1\alpha_2 - \alpha_2\alpha_3)(1 - 2\alpha_1\alpha_3) + \\
&\quad (1 - \alpha_1\alpha_3 - \alpha_2\alpha_3)(1 - 2\alpha_1\alpha_2) \\
&= 4(\alpha_1 + \alpha_2 + \alpha_3)\alpha_1\alpha_2\alpha_3 - 4(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) + 3.
\end{aligned}$$

Hence condition (iii) is equivalent to

$$(iii') \quad 2\gamma_1\gamma_2 + 2\gamma_1\gamma_3 + 2\gamma_2\gamma_3 - \gamma_1^2 - \gamma_2^2 - \gamma_3^2 \geq 0.$$

If we assume (ii), i.e. $\gamma_i \geq 0$, $i = 1, 2, 3$, the inequality (iii') can be further reduced. Indeed (iii') is equivalent to

$$\gamma_3^2 - 2(\gamma_1 + \gamma_2)\gamma_3 + (\gamma_1 - \gamma_2)^2 \leq 0.$$

The roots of this second order polynomial in γ_3 are $\gamma_1 + \gamma_2 \pm 2\sqrt{\gamma_1\gamma_2}$, so (iii') is equivalent to

$$\gamma_1 + \gamma_2 - 2\sqrt{\gamma_1\gamma_2} \leq \gamma_3 \leq \gamma_1 + \gamma_2 + 2\sqrt{\gamma_1\gamma_2}$$

or

$$|\sqrt{\gamma_1} - \sqrt{\gamma_2}| \leq \sqrt{\gamma_3} \leq \sqrt{\gamma_1} + \sqrt{\gamma_2}.$$

Hence if we assume (ii), and define the numbers δ_1, δ_2 and δ_3 by

$$\delta_i = \sqrt{\alpha_i^2 - \lambda\alpha_i + 1}, \quad i = 1, 2, 3,$$

the inequality (iii) is equivalent to the statement, that the numbers δ_1, δ_2 and δ_3 satisfy the triangle inequality,

$$|\delta_1 - \delta_2| \leq \delta_3 \leq \delta_1 + \delta_2. \quad (6.1)$$

Remark 6.3 If $\lambda \leq 2$ (ii) is trivially true, because

$$\alpha_i^2 - \lambda\alpha_i + 1 \geq \alpha_i^2 - 2\alpha_i + 1 = (\alpha_i - 1)^2.$$

Moreover for $\lambda \leq 2$, (i) \Rightarrow (iii). Indeed, if (i) holds then

$$(\lambda\alpha_1 - 1)(\lambda\alpha_2 - 1)(\lambda\alpha_3 - 1) \geq 0, \quad (6.2)$$

but the left-hand side of this inequality is equal to

$$\lambda^3\alpha_1\alpha_2\alpha_3 - \lambda^2(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) + \lambda^2 - 1,$$

so (6.2) is equivalent to

$$\lambda\alpha_1\alpha_2\alpha_3 - (\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) + 1 - \frac{1}{\lambda^2} \geq 0.$$

If $\lambda \leq 2$, then $1 - \frac{1}{\lambda^2} \leq \frac{3}{4}$, so

$$\lambda\alpha_1\alpha_2\alpha_3 - (\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) + \frac{3}{4} \geq 0,$$

which proves (iii).

The above remark shows, that for $\lambda \leq 2$ the conditions (i), (ii) and (iii) reduce to (i). (i) is clearly a necessary condition, because $\|e_{ii}\|^2 = (\lambda - \alpha_i)(\lambda \alpha_i - 1)$, and $\lambda - \alpha_i > 0$.

The following example shows, that for $\lambda \leq 2$ condition (i) is also sufficient.

Example 6.4 Assume $\lambda \leq 2$ and write $\lambda = 2 \cos \theta$, where $0 \leq \theta < \frac{\pi}{2}$. Let \mathcal{S} be the two dimensional subspace of \mathbb{C}^3 given by

$$\mathcal{S} = \left\{ (x_1, x_2, x_3) \in \mathbb{C}^3 \mid \sqrt{\alpha_1}x_1 + \sqrt{\alpha_2}x_2 + \sqrt{\alpha_3}x_3 = 0 \right\},$$

and consider the following 9 vectors in \mathbb{C}^3

$$\begin{aligned} e_{11} &= \sqrt{\frac{\lambda\alpha_1-1}{\lambda}} \begin{pmatrix} \lambda - \alpha_1 \\ -\sqrt{\alpha_1\alpha_2} \\ -\sqrt{\alpha_1\alpha_3} \end{pmatrix} & e_{12} &= \begin{pmatrix} \sqrt{\alpha_2}(\alpha_1 - e^{i\theta}) \\ \sqrt{\alpha_1}(\alpha_2 - e^{-i\theta}) \\ \sqrt{\alpha_1\alpha_2\alpha_3} \end{pmatrix} & e_{13} &= \begin{pmatrix} \sqrt{\alpha_3}(\alpha_1 - e^{i\theta}) \\ \sqrt{\alpha_1\alpha_2\alpha_3} \\ \sqrt{\alpha_1}(\alpha_3 - e^{-i\theta}) \end{pmatrix} \\ e_{21} &= \begin{pmatrix} \sqrt{\alpha_2}(\alpha_1 - e^{-i\theta}) \\ \sqrt{\alpha_1}(\alpha_2 - e^{i\theta}) \\ \sqrt{\alpha_1\alpha_2\alpha_3} \end{pmatrix} & e_{22} &= \sqrt{\frac{\lambda\alpha_2-1}{\lambda}} \begin{pmatrix} -\sqrt{\alpha_1\alpha_2} \\ \lambda - \alpha_2 \\ -\sqrt{\alpha_2\alpha_3} \end{pmatrix} & e_{23} &= \begin{pmatrix} \sqrt{\alpha_1\alpha_2\alpha_3} \\ \sqrt{\alpha_3}(\alpha_2 - e^{i\theta}) \\ \sqrt{\alpha_2}(\alpha_3 - e^{-i\theta}) \end{pmatrix} \\ e_{31} &= \begin{pmatrix} \sqrt{\alpha_3}(\alpha_1 - e^{-i\theta}) \\ \sqrt{\alpha_1\alpha_2\alpha_3} \\ \sqrt{\alpha_1}(\alpha_3 - e^{i\theta}) \end{pmatrix} & e_{32} &= \begin{pmatrix} \sqrt{\alpha_1\alpha_2\alpha_3} \\ \sqrt{\alpha_3}(\alpha_2 - e^{-i\theta}) \\ \sqrt{\alpha_2}(\alpha_3 - e^{i\theta}) \end{pmatrix} & e_{33} &= \sqrt{\frac{\lambda\alpha_3-1}{\lambda}} \begin{pmatrix} -\sqrt{\alpha_1\alpha_3} \\ -\sqrt{\alpha_2\alpha_3} \\ \lambda - \alpha_3 \end{pmatrix} \end{aligned}$$

One easily checks that $e_{ij} \in \mathcal{S}$ and that $\|e_{ij}\|^2$ is given by the formulas (a) and (b). (Note that because of the symmetry of e_{ij} , with respect to permutation of indices, it suffices to check (a) and (b) for e_{11} and e_{12} .)

To check (c) and (d) one has to identify $I_{\mathcal{S}}$ with the orthogonal projection of \mathbb{C}^3 onto \mathcal{S} . Since \mathcal{S}^\perp is spanned by the unit vector

$$\xi = \frac{1}{\sqrt{\lambda}}(\sqrt{\alpha_1}, \sqrt{\alpha_2}, \sqrt{\alpha_3})$$

one has

$$I_{\mathcal{S}} = I - \xi \otimes \bar{\xi} = \frac{1}{\lambda} \begin{pmatrix} \lambda - \alpha_1 & -\sqrt{\alpha_1\alpha_2} & -\sqrt{\alpha_1\alpha_3} \\ -\sqrt{\alpha_1\alpha_2} & \lambda - \alpha_2 & -\sqrt{\alpha_2\alpha_3} \\ -\sqrt{\alpha_1\alpha_3} & -\sqrt{\alpha_2\alpha_3} & \lambda - \alpha_3 \end{pmatrix}.$$

To check (c), for $i = 1$, we show that

$$e_{12} \otimes \bar{e}_{12} + e_{13} \otimes \bar{e}_{13} = \alpha_1 I_{\mathcal{S}} - e_{11} \otimes \bar{e}_{11}.$$

One easily gets that $e_{12} \otimes \bar{e}_{12} + e_{13} \otimes \bar{e}_{13}$ equals

$$\begin{pmatrix} (\alpha_1^2 - \lambda\alpha_1 + 1)(\lambda - \alpha_1) & -(\alpha_1^2 - \lambda\alpha_1 + 1)\sqrt{\alpha_1\alpha_2} & -(\alpha_1^2 - \lambda\alpha_1 + 1)\sqrt{\alpha_1\alpha_3} \\ -(\alpha_1^2 - \lambda\alpha_1 + 1)\sqrt{\alpha_1\alpha_2} & \alpha_1(1 - \alpha_1\alpha_2) & -\alpha_1^2\sqrt{\alpha_2\alpha_3} \\ -(\alpha_1^2 - \lambda\alpha_1 + 1)\sqrt{\alpha_1\alpha_3} & -\alpha_1^2\sqrt{\alpha_1\alpha_3} & \alpha_1(1 - \alpha_1\alpha_3) \end{pmatrix}$$

To compute $\alpha_1 I_S - e_{11} \otimes \bar{e}_{11}$, it is convenient to introduce the vector

$$e_{11}^\perp = \sqrt{\lambda\alpha_1 - 1} \begin{pmatrix} 0 \\ \sqrt{\alpha_3} \\ -\sqrt{\alpha_2} \end{pmatrix},$$

which is contained in \mathcal{S} and is orthogonal to e_{11} .

Moreover

$$\|e_{11}^\perp\|^2 = (\lambda\alpha_1 - 1)(\lambda - \alpha_1) = \|e_{11}\|^2.$$

Therefore

$$e_{11} \otimes \bar{e}_{11} + e_{11}^\perp \otimes \bar{e}_{11}^\perp = (\lambda\alpha_1 - 1)(\lambda - \alpha_1)I_S.$$

Thus

$$\alpha_1 I_S - e_{11} \otimes e_{11} = e_{11}^\perp \otimes \bar{e}_{11}^\perp + \lambda(\alpha_1^2 - \lambda\alpha_1 + 1)I_S$$

which, by straight forward computations, coincides with $e_{12} \otimes \bar{e}_{12} + e_{13} \otimes \bar{e}_{13}$ computed above. This proves (c) for $i = 1$. By symmetry it also holds for $i = 2$ and $i = 3$. Furthermore (d) follows from (c) because $\bar{e}_{ij} = e_{ji}$, $i, j = 1, 2, 3$.

Lemma 6.5 *Let x_1, x_2 and x_3 be 3 vectors in a two dimensional Hilbert space \mathcal{H} , and $c \in \mathbb{R}_+$ be such that*

$$x_1 \otimes \bar{x}_1 + x_2 \otimes \bar{x}_2 + x_3 \otimes \bar{x}_3 = cI_{\mathcal{H}}. \quad (6.3)$$

Then

- (a) $\|x_1\|^2 + \|x_2\|^2 + \|x_3\|^2 = 2c$
- (b) $\|x_i\|^2 \leq c, \quad i = 1, 2, 3.$
- (c) $|(x_i, x_j)|^2 = (c - \|x_i\|^2)(c - \|x_j\|^2), \quad i \neq j$

Proof (a) follows since $\text{Tr}(x_i \otimes \bar{x}_i) = \|x_i\|^2$ and $\text{Tr}(I_{\mathcal{H}}) = \dim \mathcal{H} = 2$. (b) is clear, because $x_i \otimes \bar{x}_i \leq I_{\mathcal{H}}$.

(c) : Let $x_i = (x_{i1}, x_{i2})$, $i = 1, 2, 3$ be the coordinates of x_i , with respect to an orthonormal basis. Define the vectors $\xi_1 = (x_{11}, x_{21}, x_{31})$ and $\xi_2 = (x_{12}, x_{22}, x_{32})$, then condition 6.3 is equivalent to ξ_1, ξ_2 being orthogonal in \mathbb{C}^3 , with $\|\xi_1\|^2 = \|\xi_2\|^2 = c$. Choose a third vector $\xi_3 = (x_{13}, x_{23}, x_{33})$ such that $\frac{1}{\sqrt{c}}\xi_1, \frac{1}{\sqrt{c}}\xi_2$ and $\frac{1}{\sqrt{c}}\xi_3$ form an orthonormal basis for \mathbb{C}^3 . Then

$$\frac{1}{\sqrt{c}} \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$$

is a unitary matrix. In particular the rows of the matrix form an orthonormal basis. I.e.

$$\begin{aligned} \|x_i\|^2 + |x_{3i}|^2 &= c, \quad i = 1, 2, 3 \\ (x_i, x_j) + x_{3i}\bar{x}_{3j} &= 0 \quad i \neq j \end{aligned}$$

Thus

$$|(x_i, x_j)| = |x_{3i}| |x_{3j}| = \sqrt{c - \|x_i\|^2} \sqrt{c - \|x_j\|^2}, \quad i \neq j,$$

which proves (c). □

Lemma 6.6 *If two vectors x_1 and x_2 in a two dimensional Hilbert space, \mathcal{H} , and $c \in \mathbb{R}_+$ are such that*

$$(a) \quad \|x_i\|^2 \leq c, \quad i = 1, 2$$

$$(b) \quad |(x_1, x_2)|^2 = (c - \|x_1\|^2)(c - \|x_2\|^2).$$

Then there exists a third vector $x_3 \in \mathcal{H}$ such that

$$x_1 \otimes \bar{x}_1 + x_2 \otimes \bar{x}_2 + x_3 \otimes \bar{x}_3 = cI_{\mathcal{H}}.$$

Moreover $\|x_3\|^2 = 2c - \|x_1\|^2 - \|x_2\|^2$.

Proof Let $x_1 = (x_{11}, x_{12})$ and $x_2 = (x_{21}, x_{22})$ be the coordinates of x_1 and x_2 in a fixed orthonormal basis. Choose $\theta \in \mathbb{R}$, such that

$$(x_1, x_2) = e^{i\theta} |(x_1, x_2)|.$$

Using (a) we can define $x_{13}, x_{23} \in \mathbb{C}$ by

$$x_{13} = e^{i\theta} \sqrt{c - \|x_1\|^2} \text{ and } x_{23} = -\sqrt{c - \|x_2\|^2}.$$

Then $\|x_i\|^2 + |x_{i3}|^2 = c$, $i = 1, 2$ and $(x_1, x_2) + x_{13}\bar{x}_{23} = 0$.

Therefore $\tilde{x}_i = (x_{i1}, x_{i2}, x_{i3})$, $i = 1, 2$ are two orthogonal vectors in \mathbb{C}^3 , both of length \sqrt{c} . Choose $\tilde{x}_3 = (x_{31}, x_{32}, x_{33})$ such that

$$\frac{1}{\sqrt{c}}\tilde{x}_1, \frac{1}{\sqrt{c}}\tilde{x}_2, \frac{1}{\sqrt{c}}\tilde{x}_3$$

form an orthonormal basis of \mathbb{C}^3 . Put $x_3 = (x_{31}, x_{32})$. Since

$$\left(\frac{1}{\sqrt{c}} x_{ij} \right)_{i,j=1}^3$$

is a unitary matrix, the two first columns are orthogonal vectors. Hence

$$x_1 \otimes \bar{x}_1 + x_2 \otimes \bar{x}_2 + x_3 \otimes \bar{x}_3 = cI_{\mathcal{H}},$$

and by lemma 6.5 (a) we have

$$\|x_3\|^2 = 2c - \|x_1\|^2 - \|x_2\|^2.$$

□

Proposition 6.7 *Condition (1) of Theorem 6.1 is equivalent to*

(1') *There exists 6 vectors $(e_{ij})_{i,j=1}^3$, $i \neq j$, in a two dimensional Hilbert space, such that*

$$(e) \quad \|e_{ij}\|^2 = \alpha_i + \alpha_j - \lambda \alpha_i \alpha_j, \quad i \neq j$$

$$(f) \quad |(e_{ij} e_{ik})| = \sqrt{\alpha_j \alpha_k} (\lambda \alpha_i - 1), \quad i \neq j \neq k \neq i$$

$$(g) \quad e_{12} \otimes \bar{e}_{12} - e_{21} \otimes \bar{e}_{21} = e_{23} \otimes \bar{e}_{23} - e_{32} \otimes \bar{e}_{32} = e_{31} \otimes \bar{e}_{31} - e_{13} \otimes \bar{e}_{13}.$$

Proof

(1) \Rightarrow (1'): Let $(e_{ij})_{i,j=1}^3$ be as in (1) of Theorem 6.1, and consider the six vectors corresponding to $i \neq j$. Then (e) holds. From (c) in Theorem 6.1 we have

$$e_{ii} \otimes \bar{e}_{ii} + e_{ij} \otimes \bar{e}_{ij} + e_{ik} \otimes \bar{e}_{ik} = \alpha_i I,$$

so by lemma 6.5 (c) we have

$$\begin{aligned} |(e_{ij}, e_{ik})|^2 &= (\alpha_i - \|e_{ij}\|^2)(\alpha_i - \|e_{ik}\|^2) \\ &= \alpha_j \alpha_k (\lambda \alpha_i + 1)^2. \end{aligned}$$

This proves (f), because $\lambda \alpha_i - 1 \geq 0$ by (a) of Theorem 6.1.

By (c) and (d) of Theorem 6.1 we have

$$\sum_{j=1}^3 (e_{ij} \otimes \bar{e}_{ij} - e_{ji} \otimes \bar{e}_{ji}) = 0.$$

The term with $j = i$ vanishes in the sum, and by rearranging the remaining terms one gets (g).

(1') \Rightarrow (1): Assume $(e_{ij})_{i,j=1,i \neq j}^3$ satisfy (1'). Condition (g) can be rewritten as

$$\sum_{j \neq i} e_{ij} \otimes \bar{e}_{ij} = \sum_{j \neq i} e_{ji} \otimes \bar{e}_{ji}, \quad i = 1, 2, 3. \quad (6.4)$$

Fix i , and let j, k be the two remaining numbers in $\{1, 2, 3\}$. By (f) $\lambda \alpha_i - 1 \geq 0$, so by (e)

$$\begin{aligned} \alpha_i - \|e_{ij}\|^2 &= \alpha_j (\lambda \alpha_i - 1) \geq 0 \\ \alpha_i - \|e_{ik}\|^2 &= \alpha_k (\lambda \alpha_i - 1) \geq 0. \end{aligned}$$

Moreover, by (e) and (f)

$$|(e_{ij}, e_{ik})|^2 = (\alpha_i - \|e_{ij}\|^2)(\alpha_i - \|e_{ik}\|^2).$$

Hence, by lemma 6.6, there exists $e_{ii} \in \mathcal{H}$, such that

$$\sum_{l=1}^3 e_{il} \otimes \bar{e}_{il} = \alpha_i I_{\mathcal{H}},$$

and

$$\begin{aligned} \|e_{ii}\|^2 &= 2\alpha_i - \|e_{ij}\|^2 - \|e_{ik}\|^2 \\ &= (\alpha_j + \alpha_k)(\lambda \alpha_i - 1) \\ &= (\lambda - \alpha_i)(\lambda \alpha_i - 1). \end{aligned}$$

By (6.4) also $\sum_{l=1}^3 e_{li} \otimes \bar{e}_{li} = \alpha_i I_{\mathcal{H}}$. Hence this construction, for $i = 1, 2, 3$, provides us with three new vectors e_{11}, e_{22}, e_{33} , which, together with the given six vectors, satisfy (a), (b), (c) and (d) in Theorem 6.1. \square

Lemma 6.8 Let $A = (a_{ij})_{i,j=1}^3$ be a symmetric 3×3 matrix with non-negative entries, and set $A' = (a'_{ij})_{i,j=1}^3$, where $a'_{ii} = a_{ii}$ and $a'_{ij} = -a_{ij}$, $i \neq j$. Then the following two conditions are equivalent

(1) There exists 3 vectors ξ_1, ξ_2, ξ_3 in \mathbb{C}^3 with

$$|(\xi_i, \xi_j)| = a_{ij}, \quad i, j = 1, 2, 3$$

(2) $\det(A) \geq 0$, $\det(A') \leq 0$ and $a_{ij}^2 \leq a_{ii}a_{jj}$, $i \neq j$.

Proof

(1) \Rightarrow (2): Assume (1). Then $a_{ij}^2 \leq a_{ii}a_{jj}$ by the Cauchy-Schwartz inequality. Let B be the matrix with entries $b_{ij} = (\xi_i, \xi_j)$. Since $\dim(\mathbb{C}^3) = 3$, the vectors ξ_1, ξ_2, ξ_3 are linearly dependent, i.e. there exists complex numbers $(c_1, c_2, c_3) \neq (0, 0, 0)$ such that $\sum_j c_j \xi_j = 0$. Thus

$$\sum_j b_{ij} \bar{c}_j = (\xi_i, \sum_j c_j \xi_j) = 0.$$

Hence $\det(B) = 0$.

B is of the form

$$B = \begin{pmatrix} a_{11} & \gamma_3 a_{12} & \bar{\gamma}_2 a_{13} \\ \bar{\gamma}_3 a_{12} & a_{22} & \gamma_1 a_{23} \\ \gamma_2 a_{13} & \bar{\gamma}_1 a_{23} & a_{33} \end{pmatrix},$$

where $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{C}$, $|\gamma_1| = |\gamma_2| = |\gamma_3|$.

Hence

$$\det(B) = a_{11}a_{22}a_{33} + 2\operatorname{Re}(\gamma_1\gamma_2\gamma_3)a_{12}a_{23}a_{13} - a_{11}a_{23}^2 - a_{22}a_{13}^2 - a_{33}a_{12}^2.$$

Since

$$\begin{aligned} \det(A) &= a_{11}a_{22}a_{33} + 2a_{12}a_{23}a_{13} - a_{11}a_{23}^2 - a_{22}a_{13}^2 - a_{33}a_{12}^2 \\ \det(A') &= a_{11}a_{22}a_{33} - 2a_{12}a_{23}a_{13} - a_{11}a_{23}^2 - a_{22}a_{13}^2 - a_{33}a_{12}^2 \end{aligned}$$

we have

$$\det(A') \leq \det(B) \leq \det(A)$$

which proves (2).

(2) \Rightarrow (1): Assume (2). Since $\det(A') \leq 0 \leq \det(A)$, the above formulas for $\det(A)$ and $\det(A')$, show that there exists $r \in [-1, 1]$, such that

$$a_{11}a_{22}a_{33} + 2ra_{12}a_{23}a_{13} - a_{11}a_{23}^2 - a_{22}a_{13}^2 - a_{33}a_{12}^2 = 0.$$

Let $\gamma = r + i\sqrt{1-r^2}$. Then $|\gamma| = 1$, $\operatorname{Re}(\gamma) = r$, so the matrix

$$B = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & \gamma a_{23} \\ a_{13} & \bar{\gamma} a_{23} & a_{33} \end{pmatrix}$$

has $\det(B) = 0$. Since furthermore $a_{11}, a_{22}, a_{33} \geq 0$, and the minors

$$A_{11} = \begin{pmatrix} a_{22} & \gamma a_{23} \\ \gamma a_{23} & a_{33} \end{pmatrix}, \quad A_{22} = \begin{pmatrix} a_{11} & a_{13} \\ a_{13} & a_{33} \end{pmatrix}, \quad A_{33} = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix},$$

have non-negative determinants by (2), the matrix B is positive semidefinite. (Indeed the characteristic polynomial

$$\det(\rho I - B) = \rho^3 - \rho^2(a_{11} + a_{22} + a_{33}) + \rho\left(\sum_{i=1}^3 \det(A_{ii})\right) - \det(B)$$

is strictly negative when $\rho < 0$, therefore B has only non-negative eigenvalues.)

Let ξ_1, ξ_2, ξ_3 be the column vectors of the matrix $B^{\frac{1}{2}}$. Then

$$|(\xi_i, \xi_j)| = |b_{ij}| = a_{ij}.$$

Moreover $\text{span}\{\xi_1, \xi_2, \xi_3\} = \text{range}(B^{\frac{1}{2}}) = \text{range}(B)$ has dimension at most 2, because $\det(B) = 0$. This proves (1). \square

Proof of sufficiency of (i),(ii) and (iii) in Theorem 6.1

The case $\lambda \leq 2$ was treated in example 6.4, so we can assume $\lambda > 2$. Assume (i), (ii) and (iii), and set

$$A = (a_{ij})_{i,j=1}^3 = \begin{pmatrix} \alpha_2 + \alpha_3 - \lambda \alpha_2 \alpha_3 & \sqrt{\alpha_1 \alpha_2}(\lambda \alpha_3 - 1) & \sqrt{\alpha_1 \alpha_3}(\lambda \alpha_2 - 1) \\ \sqrt{\alpha_1 \alpha_2}(\lambda \alpha_3 - 1) & \alpha_1 + \alpha_3 - \lambda \alpha_1 \alpha_3 & \sqrt{\alpha_2 \alpha_3}(\lambda \alpha_1 - 1) \\ \sqrt{\alpha_1 \alpha_3}(\lambda \alpha_2 - 1) & \sqrt{\alpha_2 \alpha_3}(\lambda \alpha_1 - 1) & \alpha_1 + \alpha_2 - \lambda \alpha_1 \alpha_2 \end{pmatrix}$$

Then A has non-negative entries, because $\lambda \alpha_i - 1 \geq 0$ by (i), and for $i \neq j$ we have

$$\alpha_i + \alpha_j - \lambda \alpha_i \alpha_j = (\lambda - \alpha_i - \alpha_j)(\lambda \alpha_i - 1) + \lambda(\alpha_i^2 - \lambda \alpha_i + 1) \geq 0$$

by (i) and (ii). We will show, that A satisfies condition (2) of lemma 6.8. Let A' be as in lemma 6.8. A tedious, but straight forward computation shows that

$$\begin{aligned} \det(A) &= \lambda^2 \alpha_1 \alpha_2 \alpha_3 (4\lambda \alpha_1 \alpha_2 \alpha_3 - 4(\alpha_1 \alpha_2 + \alpha_1 \alpha_3 + \alpha_2 \alpha_3) + 3) \\ \det(A') &= \lambda(4 - \lambda^2) \alpha_1 \alpha_2 \alpha_3. \end{aligned}$$

Hence $\det(A) \geq 0$ by (iii) and $\det(A') < 0$, because $\lambda > 2$.

Moreover by (ii)

$$\begin{aligned} a_{11} a_{22} - a_{12}^2 &= a_{11} a_{22} - (\alpha_3 - a_{11})(\alpha_3 - a_{22}) \\ &= \alpha_3(a_{11} + a_{22} - \alpha_3) \\ &= \lambda \alpha_3(\lambda^2 - \lambda \alpha_3 + 1) \\ &\geq 0, \end{aligned}$$

and similarly $a_{11}a_{33} - a_{13}^2 \geq 0$ and $a_{22}a_{33} - a_{23}^2 \geq 0$. Hence by lemma 6.8, there exists 3 vectors ξ_1, ξ_2, ξ_3 in \mathbb{C}^2 such that

$$|(\xi_i, \xi_j)| = a_{ij}, \quad i, j = 1, 2, 3.$$

Now put $e_{12} = e_{21} = \xi_3$, $e_{13} = e_{31} = \xi_2$ and $e_{23} = e_{32} = \xi_1$. Then $(e_{ij})_{i,j=1,i \neq j}^3$ clearly satisfy the conditions (e), (f) and (g) in Proposition 6.7, so it can be extended to a set of 9 vectors $(e_{ij})_{i,j=1}^3$ satisfying (a), (b), (c) and (d) in Theorem 6.1. \square

Remark 6.9 *The above solution to (a), (b) and (c) in Theorem 6.1 for $\lambda > 2$ satisfies $e_{ij} = e_{ji}$, $i, j = 1, 2, 3$. However if $\lambda < 2$, $\det(A') < 0$, so by lemma 6.8 there are no solutions to (a), (b), (c) and (d) which satisfy $e_{ij} = e_{ji}$, $i, j = 1, 2, 3$.*

To prove that (i) (ii) and (iii) are necessary conditions, we need to look at the following map:

Lemma 6.10 *The map*

$$q : \mathbb{C}^2 \rightarrow \{a \in M_2(\mathbb{C}) | a = a^*, \text{Tr}(a) = 0\}$$

given by

$$q(x) = \sqrt{2}(x \otimes \bar{x} - \frac{1}{2}\|x\|^2 I)$$

has the following properties

- (1) $q(x) = q(y)$ if and only if $y = \gamma x$ for some $\gamma \in \mathbb{C}$, $|\gamma| = 1$.
- (2) q maps \mathbb{C}^2 onto $\{a \in M_2(\mathbb{C}) | a = a^*, \text{Tr}(a) = 0\}$.
- (3) $\|q(x)\| = \|x\|^2$, where we consider the Hilbert-Schmidt norm on $M_2(\mathbb{C})$.
- (4) $(q(x), q(y)) = 2|(x, y)|^2 - \|x\|^2\|y\|^2$, where $(q(x), q(y)) = \text{tr}(q(x)q(y))$.

Remark 6.11 By (3) and (4) it follows, that if $(x, y) = \|x\|\|y\|\cos \theta$, $0 \leq \theta \leq \frac{\pi}{2}$, then $(q(x), q(y)) = \|q(x)\|\|q(y)\|\cos 2\theta$. Hence q doubles the angles between vectors.

Proof of lemma 6.10 Since $\frac{1}{2}\|x\|^2 I$ is equal to the orthogonal projection of $x \otimes \bar{x}$ onto $\mathbb{R}I$ in $\{a \in M_2(\mathbb{C}) | a = a^*\}$, with respect to the inner product $(a, b) = \text{Tr}(ab)$, we have

$$\begin{aligned} |(x, y)|^2 &= (x \otimes \bar{x}, y \otimes \bar{y}) \\ &= (\frac{1}{2}\|x\|^2 I, \frac{1}{2}\|y\|^2 I) + (x \otimes \bar{x} - \frac{1}{2}\|x\|^2 I, y \otimes \bar{y} - \frac{1}{2}\|y\|^2 I) \\ &= \frac{1}{4}\|x\|^2\|y\|^2 \text{Tr}(I) + \frac{1}{2}(q(x), q(y)). \end{aligned}$$

Since $\text{Tr}(I) = 2$ (I is the identity on \mathbb{C}^2), (4) follows. Moreover (4) \Rightarrow (3) is trivial.

By (3), $q(x) = q(y)$ implies $\|x\| = \|y\|$, and then $x \otimes \bar{x} = y \otimes \bar{y}$ by the definition of q . This shows that $y = \gamma x$ for some $\gamma \in \mathbb{C}$, $|\gamma| = 1$. Hence (1) holds.

Finally if $a \in M_2(\mathbb{C})$, $a = a^*$, $\text{Tr}(a) = 0$, then the eigenvalues of a are $\{\rho, -\rho\}$ for some $\rho \geq 0$, and we can choose an orthonormal basis (e_1, e_2) for \mathbb{C}^2 , such that

$$\begin{aligned} a &= \rho e_1 \otimes \bar{e}_1 - \rho e_2 \otimes \bar{e}_2 \\ &= 2\rho e_1 \otimes \bar{e}_1 - \rho I, \end{aligned}$$

since $e_1 \otimes \bar{e}_1 + e_2 \otimes \bar{e}_2 = I$. Hence $a = q(x)$, where $x = (\sqrt{2}\rho)^{\frac{1}{2}} e_1$. This shows (2). \square

Lemma 6.12 *Let $A = (a_{ij})_{i,j=1}^3$ be a matrix with non-negative entries, and define a 3×3 matrix $Q(A)$ by*

$$Q(A)_{ij} = \begin{cases} a_{ii}^2, & i = j \\ 2a_{ij}^2 - a_{ii} - a_{jj}, & i \neq j \end{cases}$$

Then the conditions (1) and (2) of lemma 6.8 are equivalent to

(3) $Q(A)$ is positive semidefinite.

Proof Since $(1) \Leftrightarrow (2)$, it suffices to prove $(3) \Rightarrow (1) \Rightarrow (3)$, where (1) and (2) refer to lemma 6.8.

Assume $\xi_1, \xi_2, \xi_3 \in \mathbb{C}^2$ are vectors, such that

$$|(\xi_i, \xi_j)| = a_{ij}, \quad i, j = 1, 2, 3.$$

By lemma 6.10 (3) and (4)

$$(q(\xi_i), q(\xi_j)) = Q(A)_{ij}, \quad i, j = 1, 2, 3.$$

For $c_1, c_2, c_3 \in \mathbb{C}$,

$$\sum_{i,j} (q(\xi_i), q(\xi_j)) c_i \bar{c}_j = \left\| \sum_i c_i q(\xi_i) \right\|^2 \geq 0,$$

which shows that $Q(A)$ is positive semidefinite. Hence $(1) \Rightarrow (3)$. Assume conversely that $Q(A)$ is positive semidefinite. Let η_1, η_2, η_3 be the columns of $Q(A)^{\frac{1}{2}}$. Then $\eta_1, \eta_2, \eta_3 \in \mathbb{R}^3$ and

$$(\eta_i, \eta_j) = Q(A)_{ij}, \quad i, j = 1, 2, 3.$$

We may identify \mathbb{R}^3 with the Hilbert space

$$\{a \in M_2(\mathbb{C}) | a = a^*, \text{Tr}(a) = 0\},$$

with inner product $(a, b) = \text{Tr}(ab)$. Hence by lemma 6.10 (2) there exists $\xi_1, \xi_2, \xi_3 \in \mathbb{C}^2$ such that

$$(q(\xi_i), q(\xi_j)) = Q(A)_{ij}, \quad i, j = 1, 2, 3.$$

Thus

$$\|\xi_i\|^4 = a_{ii}^2 \text{ and } 2|(\xi_i, \xi_j)|^2 - \|\xi_i\|^2 \|\xi_j\|^2 = 2a_{ij}^2 - a_{ii}a_{jj}$$

by lemma 6.10 and the definition of $Q(A)$.

Since $a_{ij} \geq 0$ for all i, j , it follows that

$$\|\xi_i\|^2 = a_{ii} \text{ and } |(\xi_i, \xi_j)| = a_{ij}, \quad i \neq j.$$

Hence A satisfies (1) of lemma 6.8, and $(3) \Rightarrow (1)$. \square

Lemma 6.13 Let $\alpha_1, \alpha_2, \alpha_3 > 0$ and $\lambda = \alpha_1 + \alpha_2 + \alpha_3$. Put

$$A = (a_{ij})_{i,j=1}^3 = \begin{pmatrix} \alpha_2 + \alpha_3 - \lambda\alpha_2\alpha_3 & \sqrt{\alpha_1\alpha_2}(\lambda\alpha_3 - 1) & \sqrt{\alpha_1\alpha_3}(\lambda\alpha_2 - 1) \\ \sqrt{\alpha_1\alpha_2}(\lambda\alpha_3 - 1) & \alpha_1 + \alpha_3 - \lambda\alpha_1\alpha_3 & \sqrt{\alpha_2\alpha_3}(\lambda\alpha_1 - 1) \\ \sqrt{\alpha_1\alpha_3}(\lambda\alpha_2 - 1) & \sqrt{\alpha_2\alpha_3}(\lambda\alpha_1 - 1) & \alpha_1 + \alpha_2 - \lambda\alpha_1\alpha_2 \end{pmatrix},$$

as in the proof of sufficiency of (i), (ii) and (iii) in Theorem 6.1, and let $Q(A)$ be the matrix with entries

$$Q(A)_{ij} = \begin{cases} a_{ii}^2, & i = j \\ 2a_{ij}^2 - a_{ii}a_{jj}, & i \neq j. \end{cases}$$

Moreover set

$$D = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}.$$

Then condition (1) in Theorem 6.1 implies that $a_{ij} \geq 0$, $i, j = 1, 2, 3$, and that

$$Q(A) - tD$$

is positive semidefinite for some $t \geq 0$.

Proof Clearly $a_{ij} \geq 0$, by (a) and (b) of Theorem 6.1. Let $(e_{ij})_{i,j=1}^3$ be a solution to (a), (b), (c) and (d) of Theorem 6.1. Set

$$q_{ij} = q(e_{ij}) = \sqrt{2}(e_{ij} \otimes \bar{e}_{ij} - \frac{1}{2}\|e_{ij}\|^2).$$

By Proposition 6.7 and lemma 6.10

$$(i) \quad \|q_{ij}\| = \|e_{ij}\|^2.$$

$$(ii) \quad (q_{ij}, q_{ik}) = 2|(e_{ij}, e_{ik})|^2 - \|e_{ij}\|^2\|e_{ik}\|^2, \quad i \neq j \neq k \neq i.$$

$$(iii) \quad q_{12} - q_{21} = q_{31} - q_{13} = q_{23} - q_{32}.$$

Put $q_{ij}^s = \frac{1}{2}(q_{ij} + q_{ji})$, $i \neq j$ and

$$q^a = \frac{1}{2}(q_{12} - q_{21}) = \frac{1}{2}(q_{23} - q_{32}) = \frac{1}{2}(q_{31} - q_{13}).$$

Then

$$\begin{aligned} q_{12} &= q_{12}^s + q^a, & q_{21} &= q_{21}^s - q^a \\ q_{23} &= q_{23}^s + q^a, & q_{32} &= q_{32}^s - q^a \\ q_{31} &= q_{31}^s + q^a, & q_{13} &= q_{13}^s - q^a. \end{aligned}$$

Since $\|q_{ij}\|^2 = \|q_{ji}\|^2$ it follows that q^a is orthogonal to q_{12}^s, q_{23}^s and q_{31}^s . (The range of q is a real Hilbert space.) Set $t = \|q^a\|^2$. Then

$$\|q_{ij}\|^2 = \|q_{ij}^s\|^2 + t, \quad i \neq j.$$

Moreover

$$\begin{aligned}(q_{12}, q_{31}) &= (q_{12}^s, q_{31}^s) - t \\ (q_{21}, q_{23}) &= (q_{21}^s, q_{23}^s) - t \\ (q_{31}, q_{32}) &= (q_{31}^s, q_{32}^s) - t.\end{aligned}$$

Hence by (i) and (ii)

$$\begin{aligned}(iv) \quad \|q_{ij}^s\|^2 &= \|e_{ij}\|^4 - t, \quad i \neq j. \\ (v) \quad (q_{12}^s, q_{31}^s) &= 2|(e_{12}, e_{31})|^2 - \|e_{12}\|^2 \|e_{31}\|^2 + t. \\ (vi) \quad (q_{21}^s, q_{23}^s) &= 2|(e_{21}, e_{23})|^2 - \|e_{21}\|^2 \|e_{23}\|^2 + t. \\ (vii) \quad (q_{31}^s, q_{32}^s) &= 2|(e_{31}, e_{32})|^2 - \|e_{31}\|^2 \|e_{32}\|^2 + t.\end{aligned}$$

Put $\eta_1 = q_{23}^s, \eta_2 = q_{31}^s$ and $\eta_3 = q_{12}^s$. By (a) and (b) of Theorem 6.1 and the definition of A we have

$$\begin{aligned}a_{11} &= \|e_{23}\|^2 = \|e_{32}\|^2, & a_{12} &= a_{21} = |(e_{31}, e_{32})|, \\ a_{22} &= \|e_{13}\|^2 = \|e_{31}\|^2, & a_{13} &= a_{31} = |(e_{21}, e_{23})|, \\ a_{33} &= \|e_{12}\|^2 = \|e_{21}\|^2, & a_{23} &= a_{32} = |(e_{12}, e_{13})|.\end{aligned}$$

Hence by the definition of $Q(A)$ and (iv), (v), (vi) and (vii),

$$((\eta_i, \eta_j))_{i,j=1}^3 = Q(A) - tD.$$

For $(c_1, c_2, c_3) \in \mathbb{C}^3$,

$$\sum_{i=1}^3 (\eta_i, \eta_j) c_i \bar{c}_j = \left\| \sum_{i=1}^3 c_i \eta_i \right\|^2 \geq 0,$$

so the matrix is positive semidefinite. This proves lemma 6.13. \square

Lemma 6.14 *Let $\alpha_1, \alpha_2, \alpha_3 > 0$ and $\lambda = \alpha_1 + \alpha_2 + \alpha_3$. Let $A, Q(A)$ and D be as in lemma 6.13. Then*

$$Q(A) - t_0 D$$

is positive semidefinite for $t_0 = \lambda(4 - \lambda^2)\alpha_1\alpha_2\alpha_3$.

Proof Assume first that $\lambda \leq 2$ and $\lambda\alpha_i - 1 \geq 0$, $i = 1, 2, 3$. Let $(e_{ij})_{i,j=1}^3$ be the explicit solution to condition (1) in Theorem 6.1, described in example 6.4. By the proof of lemma 6.13, $Q(A) + tD$ is positive definite for

$$\begin{aligned}t &= \left\| \frac{1}{2}(q(e_{12}) - q(e_{21})) \right\|^2 \\ &= \left\| \frac{1}{\sqrt{2}}(e_{12} \otimes \bar{e}_{12} - e_{21} \otimes \bar{e}_{21}) \right\|^2.\end{aligned}$$

With the notation of example 6.4 we have

$$e_{12} \otimes \bar{e}_{12} - e_{21} \otimes \bar{e}_{21} = i2 \sin(\theta) \sqrt{\alpha_1 \alpha_2 \alpha_3} \begin{pmatrix} 0 & \sqrt{\alpha_3} & -\sqrt{\alpha_2} \\ -\sqrt{\alpha_3} & 0 & \sqrt{\alpha_1} \\ \sqrt{\alpha_2} & -\sqrt{\alpha_1} & 0 \end{pmatrix}.$$

Hence

$$t = 4\lambda \sin^2(\theta) \alpha_1 \alpha_2 \alpha_3 = \lambda(4 - \lambda^2) \alpha_1 \alpha_2 \alpha_3 = t_0.$$

Furthermore by the proof of lemma 6.12

$$Q(A) - t_0 D = ((\eta_i, \eta_j))_{i,j=1}^3,$$

where

$$\begin{aligned} \eta_1 &= \frac{1}{2}(q(e_{23}) + q(e_{32})) \\ \eta_2 &= \frac{1}{2}(q(e_{13}) + q(e_{31})) \\ \eta_3 &= \frac{1}{2}(q(e_{12}) + q(e_{21})). \end{aligned}$$

By (c) and (d) of Theorem 6.1

$$\sum_{i=1}^3 (q(e_{ij}) + q(e_{ji}))$$

is a multiple of I_S , but since $\text{Tr}(q(x)) = 0$ for all $x \in \mathcal{S}$ actually

$$\sum_{i=1}^3 (q(e_{ij}) + q(e_{ji})) = 0, \quad j = 1, 2, 3.$$

Hence $\eta_2 + \eta_3 = -q(e_{11})$, $\eta_1 + \eta_3 = -q(e_{22})$ and $\eta_1 + \eta_2 = -q(e_{33})$ which gives

$$\begin{aligned} \eta_1 &= q(e_{11}) - q(e_{22}) - q(e_{33}) \\ \eta_2 &= q(e_{22}) - q(e_{11}) - q(e_{33}) \\ \eta_3 &= q(e_{33}) - q(e_{11}) - q(e_{22}). \end{aligned}$$

But $q(e_{ii})$, $i = 1, 2, 3$ are given by functions of $\alpha_1, \alpha_2, \alpha_3$ (and $\lambda = \alpha_1 + \alpha_2 + \alpha_3$) which are real analytic in the region $\alpha_1 > 0, \alpha_2 > 0$, and $\alpha_3 > 0$. For instance

$$\begin{aligned} q(e_{11}) &= \sqrt{2}(e_{11} \otimes \bar{e}_{11} - \frac{1}{2}\|e_{11}\|^2 I_S) \\ &= \frac{\sqrt{2}(\lambda\alpha_1 - 1)}{\lambda} \begin{pmatrix} (\lambda - \alpha_1)^2 & -\sqrt{\alpha_1\alpha_2}(\lambda - \alpha_1) & -\sqrt{\alpha_1\alpha_2}(\lambda - \alpha_1) \\ -\sqrt{\alpha_1\alpha_2}(\lambda - \alpha_1) & \alpha_1\alpha_2 & \alpha_1\sqrt{\alpha_2\alpha_3} \\ -\sqrt{\alpha_1\alpha_2}(\lambda - \alpha_1) & \alpha_1\sqrt{\alpha_2\alpha_3} & \alpha_2\alpha_3 \end{pmatrix} \\ &\quad - \frac{(\lambda - \alpha_1)(\lambda\alpha_1 - 1)}{\sqrt{2}\lambda} \begin{pmatrix} \lambda - \alpha_1 & -\sqrt{\alpha_1\alpha_2} & -\sqrt{\alpha_1\alpha_3} \\ -\sqrt{\alpha_1\alpha_2} & \lambda - \alpha_2 & -\sqrt{\alpha_2\alpha_3} \\ -\sqrt{\alpha_1\alpha_3} & -\sqrt{\alpha_2\alpha_3} & \lambda - \alpha_3 \end{pmatrix}. \end{aligned}$$

Hence η_1, η_2, η_3 can also be extended to real analytic functions (with values in $B(\mathcal{S})$ identified with a subspace of $M_3(\mathbb{C})$). Since the entries of $Q(A) - t_0 D$ are polynomials in $\alpha_1, \alpha_2, \alpha_3$, it follows by uniqueness of analytic continuation, that

$$Q(A) - t_0 D = ((\eta_i, \eta_j))_{i,j=1}^3,$$

for all $(\alpha_1, \alpha_2, \alpha_3)$, $\alpha_i > 0$, $i = 1, 2, 3$, and therefore $Q(A) - t_0 D$ is positive semidefinite. \square

Proof of necessity of (i), (ii) and (ii) in Theorem 6.1 By remark 6.2, (i) and (ii) are necessary. If $\lambda \leq 2$ (i) \Rightarrow (iii) by remark 6.3, so in this case (iii) is a necessary (but redundant!) condition.

Assume $\lambda > 2$. If $(e_{ij})_{i,j=1}^3$ satisfy (1) in Theorem 6.1, then by lemma 6.13 and lemma 6.14

$$Q(A) - tD$$

is positive semidefinite for some $t \geq 0$, and also for $t = t_0 = \lambda(4 - \lambda^2)\alpha_1\alpha_2\alpha_3 < 0$, so by convexity of the positive cone in $M_3(\mathbb{R})$, $Q(A)$ is also positive semidefinite. Hence, by lemma 6.12 and lemma 6.8, $\det(A) \geq 0$. We have previously found that

$$\det(A) = \lambda^2\alpha_1\alpha_2\alpha_3(4\lambda\alpha_1\alpha_2\alpha_3 - 4(\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3) + 3).$$

Thus (1) implies (iii) in Theorem 6.1. This completes the proof of Theorem 6.1. \square

7 The Graphs Satisfying the Condition

In this section we will determine which of the graphs

$$\Gamma = S(i, j, k), \quad i \leq j \leq k, \quad \text{with } \lambda(\Gamma) \geq 2$$

satisfy the condition (6.1). The two other conditions of theorem 6.1 are trivially satisfied by the α 's coming from $S(i, j, k)$. If λ_0 denotes the Perron-Frobenius eigenvalue of the adjacency matrix of $S(i, j, k)$, $i \leq j \leq k$ then the condition is

$$\frac{1}{R_i(\lambda_0)} \leq \frac{1}{R_j(\lambda_0)} + \frac{1}{R_k(\lambda_0)},$$

or, if we multiply by $R_i(\lambda_0)$

$$1 \leq \frac{R_i(\lambda_0)}{R_j(\lambda_0)} + \frac{R_i(\lambda_0)}{R_k(\lambda_0)}. \quad (7.1)$$

Lemma 7.1 *For $m \geq 1$, $n \geq 0$, and $\lambda \geq 2$, $\frac{R_n(\lambda)}{R_{n+m}(\lambda)}$ has the following properties*

1. $\frac{R_n(\lambda)}{R_{n+m}(\lambda)}$ is decreasing in λ .
2. $\frac{R_n(\lambda)}{R_{n+m}(\lambda)}$ is increasing in n .
3. $\frac{R_n(\lambda)}{R_{n+m}(\lambda)}$ is decreasing in m .
4. $\lim_{n \rightarrow \infty} \frac{R_n(\lambda)}{R_{n+m}(\lambda)} = e^{-m\lambda}$.

Proof We refer to the proof of corollary 3.2 and remark 3.3. \square

Lemma 7.2

1. For $\lambda_0 = \lambda(S(j, j+1, j+1))$ we have $\frac{R_j(\lambda_0)}{R_{j+1}(\lambda_0)} = \frac{1}{\sqrt{2}}$.
2. For $\lambda_0 = \lambda(S(j, j+2, j+2))$ we have $\frac{R_j(\lambda_0)}{R_{j+2}(\lambda_0)} = \frac{1}{2}$.

Proof 1. λ_0 satisfies the equation

$$\begin{aligned}
\lambda_0 &= \frac{R_{j-1}(\lambda_0)}{R_j(\lambda_0)} + 2 \frac{R_j(\lambda_0)}{R_{j+1}(\lambda_0)} \\
\Downarrow \\
2 \frac{R_j(\lambda_0)}{R_{j+1}(\lambda_0)} &= \frac{\lambda_0 R_j(\lambda_0) - R_{j-1}(\lambda_0)}{R_j(\lambda_0)} \\
\Downarrow \\
2 \frac{R_j(\lambda_0)}{R_{j+1}(\lambda_0)} &= \frac{R_{j+1}(\lambda_0)}{R_j(\lambda_0)} \\
\Downarrow \\
\frac{R_j(\lambda_0)}{R_{j+1}(\lambda_0)} &= \frac{1}{\sqrt{2}}
\end{aligned}$$

2. Here λ_0 satisfies the equation

$$\begin{aligned}
\lambda_0 &= \frac{R_{j-1}(\lambda_0)}{R_j(\lambda_0)} + 2 \frac{R_{j+1}(\lambda_0)}{R_{j+2}(\lambda_0)} \\
\Downarrow \\
\frac{R_{j+1}(\lambda_0)}{R_j(\lambda_0)} &= 2 \frac{R_{j+1}(\lambda_0)}{R_{j+2}(\lambda_0)} \\
\Downarrow \\
\frac{R_j(\lambda_0)}{R_{j+2}(\lambda_0)} &= \frac{1}{2}.
\end{aligned}$$

□

Lemma 7.3 If $S(j, k, l)$, $j \leq k \leq l$ does not satisfy (7.1), then $S(j, k+k', l+l')$, $j \leq k+k' \leq l+l'$ does not satisfy (7.1).

Proof Let $\lambda_0 = \lambda(S(j, k, l))$ and $\lambda_1 = \lambda(S(j, k+k', l+l'))$. Then $\lambda_1 \geq \lambda_0$ and we have

$$\begin{aligned}
\frac{R_j(\lambda_1)}{R_{k+k'}(\lambda_1)} + \frac{R_j(\lambda_1)}{R_{l+l'}(\lambda_1)} &\leq \quad (\text{by lemma 7.1 (1)}) \\
\frac{R_j(\lambda_0)}{R_{k+k'}(\lambda_0)} + \frac{R_j(\lambda_0)}{R_{l+l'}(\lambda_0)} &\leq \quad (\text{by lemma 7.1 (2)}) \\
\frac{R_j(\lambda_0)}{R_k(\lambda_0)} + \frac{R_j(\lambda_0)}{R_l(\lambda_0)} &< 1.
\end{aligned}$$

Hence $S(j, k+k', l+l')$, $j \leq k+k' \leq l+l'$ does not satisfy (7.1). □

Proposition 7.4 $S(j, k, l)$, $j \leq k \leq l$ satisfies (7.1) if and only if (j, k, l) is of one of the following

- 1) $(j, j, j+n)$ $j \geq 2, n \geq 0$
- 2) $(j, j+1, j+1)$ $j \geq 2$
- 3) $(j, j+1, j+2)$ $j \geq 2$
- 4) $(j, j+1, j+3)$ $j \geq 2$
- 5) $(j, j+2, j+2)$ $j \geq 1$

Proof Let λ_∞ denote the largest eigenvalue of “S(∞, ∞, ∞)”. It is not difficult to show that $\lambda_\infty = \frac{3\sqrt{2}}{2}$.

1) If $\lambda_0 = \lambda(S(j, j, j + n))$, then clearly $\frac{R_j(\lambda_0)}{R_j(\lambda_0)} + \frac{R_j(\lambda_0)}{R_{j+n}(\lambda_0)} \geq 1$.

2) Let $\lambda_0 = \lambda(S(j, j + 1, j + 1))$. We must show that $2\frac{R_j(\lambda_0)}{R_{j+1}(\lambda_0)} \geq 1$. By lemma 7.1 we have

$$2\frac{R_j(\lambda_0)}{R_{j+1}(\lambda_0)} \geq 2\frac{R_1(\lambda_0)}{R_2(\lambda_0)} \geq 2\frac{R_1(\lambda_\infty)}{R_2(\lambda_\infty)} = \frac{6\sqrt{2}}{7} > 1.$$

Hence $S(j, j + 1, j + 1)$ satisfies (7.1).

3) Let $\lambda_0 = \lambda(S(j, j + 1, j + 2))$. We must show that $\frac{R_j(\lambda_0)}{R_{j+1}(\lambda_0)} + \frac{R_j(\lambda_0)}{R_{j+2}(\lambda_0)} \geq 1$. We have

$$\begin{aligned} \frac{R_j(\lambda_0)}{R_{j+1}(\lambda_0)} &\geq \frac{R_1(\lambda_0)}{R_2(\lambda_0)} \geq \frac{R_1(\lambda_\infty)}{R_2(\lambda_\infty)} = \frac{3\sqrt{2}}{7} \\ \frac{R_j(\lambda_0)}{R_{j+2}(\lambda_0)} &\geq \frac{R_1(\lambda_0)}{R_3(\lambda_0)} \geq \frac{R_1(\lambda_\infty)}{R_3(\lambda_\infty)} = \frac{2}{5}. \end{aligned}$$

So $\frac{R_j(\lambda_0)}{R_{j+1}(\lambda_0)} + \frac{R_j(\lambda_0)}{R_{j+2}(\lambda_0)} \geq \frac{3\sqrt{2}}{7} + \frac{2}{5} > 1$, and hence $S(j, j + 1, j + 2)$ satisfies (7.1).

4) Let $\lambda_0 = \lambda(S(j, j + 1, j + 3))$. We must show that $\frac{R_j(\lambda_0)}{R_{j+1}(\lambda_0)} + \frac{R_j(\lambda_0)}{R_{j+3}(\lambda_0)} \geq 1$.

For $j \geq 3$ it is enough to show

$$\frac{R_3(\lambda_\infty)}{R_4(\lambda_\infty)} + \frac{R_3(\lambda_\infty)}{R_6(\lambda_\infty)} \geq 1.$$

And since $\frac{R_3(\lambda_\infty)}{R_4(\lambda_\infty)} \approx 0.68$ and $\frac{R_3(\lambda_\infty)}{R_6(\lambda_\infty)} \approx 0.33$ this is satisfied.

For $j = 2$ we have $\lambda_0 \approx 2.0697$. Put $\lambda_1 = 2.1$, then all we need to show is

$$\frac{R_2(\lambda_1)}{R_3(\lambda_1)} + \frac{R_2(\lambda_1)}{R_5(\lambda_1)} \geq 1.$$

And since $\frac{R_2(\lambda_1)}{R_3(\lambda_1)} \approx 0.67$ and $\frac{R_2(\lambda_1)}{R_5(\lambda_1)} \approx 0.33$ this is satisfied.

5) Let $\lambda_0 = \lambda(S(j, j + 2, j + 2))$. We must show that $2\frac{R_j(\lambda_0)}{R_{j+2}(\lambda_0)} \geq 1$. λ_0 satisfies the equation

$$\begin{aligned} \lambda_0 &= \frac{R_{j-1}(\lambda_0)}{R_j(\lambda_0)} + 2\frac{R_{j+1}(\lambda_0)}{R_{j+2}(\lambda_0)} \\ \Downarrow \\ \lambda_0 R_j(\lambda_0) R_{j+2}(\lambda_0) &= R_{j-1}(\lambda_0) R_{j+2}(\lambda_0) + 2R_{j+1}(\lambda_0) R_j(\lambda_0) \\ \Downarrow \\ 2R_{j+1}(\lambda_0) R_j(\lambda_0) &= R_{j+2}(\lambda_0) (\lambda_0 R_j(\lambda_0) - R_{j-1}(\lambda_0)) \\ \Downarrow \\ 2R_{j+1}(\lambda_0) R_j(\lambda_0) &= R_{j+2}(\lambda_0) R_{j+1}(\lambda_0) \\ \Downarrow \\ 2\frac{R_j(\lambda_0)}{R_{j+2}(\lambda_0)} &= 1. \end{aligned}$$

Hence $S(j, j+2, j+2)$ satisfies (7.1).

To show that these are the only values of (j, k, l) , for which $S(j, k, l)$ satisfies (7.1), we argue as follows.

By lemma 7.3 it suffices to show that $S(j, j+1, j+4)$ and $S(j, j+2, j+3)$ do not satisfy (7.1).

Let $\lambda_0 = \lambda(S(j, j+1, j+m))$, $m \geq 1$ and $\lambda_1 = \lambda(S(j, j+1, j+1))$, then $\lambda_1 \leq \lambda_0$ and by lemmas 7.1, 7.2 we have

$$\frac{R_j(\lambda_0)}{R_{j+1}(\lambda_0)} + \frac{R_j(\lambda_0)}{R_{j+m}(\lambda_0)} \leq \frac{R_j(\lambda_1)}{R_{j+1}(\lambda_1)} + e^{-mx} = \frac{1}{\sqrt{2}} + e^{-mx}.$$

For $m = 4$, $\frac{1}{\sqrt{2}} + e^{-4x} < 1$ corresponds to $x > \frac{\ln(\sqrt{2}) - \ln(\sqrt{2}-1)}{4}$, which again corresponds to $\lambda_0^2 > 4.3889 \dots$. We have $\lambda^2(S(3, 4, 7)) \approx 4.4107$, hence we have excluded $S(j, j+1, j+m)$ for $j \geq 3$ and $m \geq 4$.

For $m = 5$, $\frac{1}{\sqrt{2}} + e^{-5x} < 1$ corresponds to $\lambda_0^2 > 4.2461 \dots$. We have $\lambda^2(S(2, 3, 7)) \approx 4.3027$, hence we have excluded $S(2, 3, 2+m)$ for $j \geq 3$ and $m \geq 5$.

We are left with the case $S(2, 3, 6)$. Let $\lambda_0 = \lambda(S(2, 3, 6)) \approx 2.0728$, and let $\lambda_1 = 2.07$. Then

$$\frac{R_2(\lambda_0)}{R_3(\lambda_0)} + \frac{R_2(\lambda_0)}{R_6(\lambda_0)} \leq \frac{R_2(\lambda_1)}{R_3(\lambda_1)} + \frac{R_2(\lambda_1)}{R_6(\lambda_1)} \approx 0.69 + 0.28 < 1.$$

Let $\lambda_0 = \lambda(S(j, j+2, j+m))$ and $\lambda_1 = \lambda(S(j, j+2, j+2))$. As before we have

$$\frac{R_j(\lambda_0)}{R_{j+2}(\lambda_0)} + \frac{R_j(\lambda_0)}{R_{j+m}(\lambda_0)} \leq \frac{R_j(\lambda_1)}{R_{j+2}(\lambda_1)} + e^{-mx} = \frac{1}{2} + e^{-mx}.$$

For $m = 3$, $\frac{1}{2} + e^{-3x} < 1$ corresponds to $x > \frac{\ln 2}{3}$, which again corresponds to $\lambda_0^2 > 4.2173 \dots$. We have $\lambda^2(S(2, 4, 5)) \approx 4.3235$, hence we have excluded $S(j, j+2, j+m)$ for $j \geq 2$ and $m \geq 3$.

For $S(1, 3, 4)$ we have $\lambda_0 \approx 2.0153$. Put $\lambda_1 = 2.01$, then

$$\frac{R_1(\lambda_0)}{R_3(\lambda_0)} + \frac{R_1(\lambda_0)}{R_4(\lambda_0)} \leq \frac{R_1(\lambda_1)}{R_3(\lambda_1)} + \frac{R_1(\lambda_1)}{R_4(\lambda_1)} \approx 0.49 + 0.38 < 1,$$

and we conclude that $S(j, j+2, j+m)$ does not satisfy (7.1) for $j \geq 1$ and $m \geq 3$. \square

The above determined graphs give rise to the following values of the index (all terms are approximate), of irreducible subfactors of the hyperfinite II_1 -factor.

Tables of the Index Values Corresponding to the Graphs:

j	S(j,j+1,j+1)	S(j,j+1,j+2)	S(j,j+1,j+3)	S(j,j+2,j+2)	j	S(j, j, ∞)
2	4.214320	4.260757	4.283998	4.302776	2	4.236068
3	4.379878	4.397514	4.406262	4.414214	3	4.382976
4	4.445787	4.453260	4.456966	4.460505	4	4.446352
5	4.474491	4.477873	4.479553	4.481194	5	4.474609
6	4.487695	4.489287	4.490080	4.490864	6	4.487721
7	4.493975	4.494744	4.495127	4.495508	7	4.493981
8	4.497024	4.497400	4.497588	4.497775	8	4.497025
9	4.498522	4.498708	4.498801	4.498894	9	4.498522
10	4.499264	4.499356	4.499402	4.499449	10	4.499264
11	4.499633	4.499679	4.499702	4.499725	11	4.499633
12	4.499817	4.499840	4.499851	4.499863	12	4.499817
13	4.499908	4.499920	4.499926	4.499931	13	4.499908
14	4.499954	4.499960	4.499963	4.499966	14	4.499954
15	4.499977	4.499980	4.499981	4.499983	15	4.499977
16	4.499989	4.499990	4.499991	4.499991	16	4.499989
17	4.499994	4.499995	4.499995	4.499996	17	4.499994
18	4.499997	4.499997	4.499998	4.499998	18	4.499997
19	4.499999	4.499999	4.499999	4.499999	19	4.499999
20	4.499999	4.499999	4.499999	4.499999	20	4.499999
21	4.500000	4.500000	4.500000	4.500000	21	4.500000

l	S(2, 2, 2 + l)	S(3, 3, 3 + l)	S(4, 4, 4 + l)	S(5, 5, 5 + l)	S(6, 6, 6 + l)	S(7, 7, 7 + l)
0	4.000000	4.302776	4.414214	4.460505	4.481194	4.490864
1	4.114908	4.342923	4.430385	4.467599	4.484472	4.492427
2	4.170086	4.362340	4.438283	4.471092	4.486095	4.493204
3	4.198691	4.372130	4.442232	4.472834	4.486905	4.493592
4	4.214320	4.377203	4.444234	4.473710	4.487311	4.493786
5	4.223177	4.379878	4.445259	4.474153	4.487514	4.493883
6	4.228328	4.381305	4.445787	4.474377	4.487617	4.493932
7	4.231379	4.382072	4.446059	4.474491	4.487669	4.493957
8	4.233210	4.382486	4.446200	4.474549	4.487695	4.493969
9	4.234318	4.382710	4.446273	4.474578	4.487708	4.493975
10	4.234993	4.382831	4.446311	4.474593	4.487714	4.493978
11	4.235407	4.382897	4.446331	4.474601	4.487718	4.493980
12	4.235660	4.382933	4.446341	4.474605	4.487719	4.493980
13	4.235817	4.382953	4.446346	4.474607	4.487720	4.493981
14	4.235913	4.382963	4.446349	4.474608	4.487721	4.493981
15	4.235972	4.382969	4.446351	4.474608	4.487721	4.493981
16	4.236009	4.382972	4.446351	4.474608	4.487721	4.493981
17	4.236031	4.382974	4.446352	4.474608	4.487721	4.493981
18	4.236045	4.382975	4.446352	4.474609	4.487721	4.493981
19	4.236054	4.382975	4.446352	4.474609	4.487721	4.493981
20	4.236059	4.382975	4.446352	4.474609	4.487721	4.493981

8 Ocneanu's example of a Commuting Square Based on the Graph $E_{10} = S(1, 2, 6)$.

The construction in this section is due to A. Ocneanu [O]. As shown in chapter I and chapter 4 it is not possible to construct commuting squares of one of the simple forms

$$\begin{array}{ccc} \mathcal{B} & \subset_{G^t} & \mathcal{D} \\ \cup_G & & \cup_{G^t} \\ \mathcal{A} & \subset_G & \mathcal{C} \end{array} \quad \begin{array}{ccc} \mathcal{B} & \subset_{G^t G - I} & \mathcal{D} \\ \cup_G & & \cup_G \\ \mathcal{A} & \subset_{G G^t - I} & \mathcal{C} \end{array},$$

where G is the inclusion matrix with Bratteli-diagram equal to E_{10} . The following example was found by a search on a computer for a polynomial P , such that $P(GG^t)$ and $P(G^t G)$ matrices with small non-negative integer entries. It turned out that

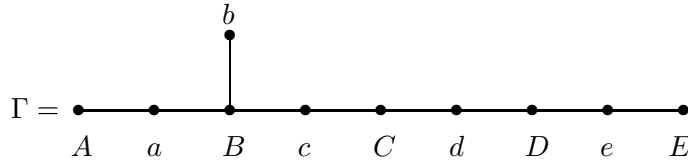
$$P(t) = t^4 - 8t^3 + 20t^2 - 16t + 3$$

could be used to produce a commuting square with inclusions

$$\begin{array}{ccc} \mathcal{C} & \subset_{P(G^t G)} & \mathcal{D} \\ \cup_G & & \cup_G \\ \mathcal{A} & \subset_{P(GG^t)} & \mathcal{B} \end{array} \quad (8.1)$$

for E_{10} . Note that $P(GG^t)G = GP(G^t G)$ and $G^t P(GG^t) = P(G^t G)G^t$, so any commuting square with these inclusion matrices will be symmetric in the sense of 1.8.

Let us label the vertices of $\Gamma = E_{10}$ as follows



where (A, B, C, D, E) and (a, b, c, d, e) correspond to the two layers in a bi-partition of Γ .

Writing the vertices in the order $(A, B, C, D, E, a, b, c, d, e)$ the adjacency matrix of Γ is

$$\Delta_\Gamma = \begin{pmatrix} 0 & G \\ G^t & 0 \end{pmatrix},$$

where

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and with the above polynomial one gets

$$P(GG^t) = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad P(G^t G) = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that $P(GG^t)$ mixes the elements of the “upper case letters” of Γ strongly, in the sense that there is an edge joining the endpoints A and E of the two long legs of the graph.

Let $c(\mathcal{A})$ (respective $c(\mathcal{B})$, $c(\mathcal{C})$ and $c(\mathcal{D})$) denote the set of minimal central projections in \mathcal{A} (respectively \mathcal{C} , \mathcal{C} and \mathcal{D}). Then, with the chosen bi-partition of E_{10} , the elements of $c(\mathcal{A})$ and $c(\mathcal{C})$ are labeled by (A, B, C, D, E) , and the elements of $c(\mathcal{B})$ and $c(\mathcal{D})$ are labeled by (a, b, c, d, e) .

To prove the existence of a commuting square of the form (8.1) is equivalent to constructing a unitary matrix, u , satisfying the bi-unitarity condition (1.9). u is of the form

$$u = \bigoplus_{(p,s)} u^{(p,s)}$$

where (p, s) runs over all $p \in c(\mathcal{A})$ and all $s \in c(\mathcal{D})$ which are connected by a path (either through $c(\mathcal{C})$ or through $c(\mathcal{B})$). Each direct summand of $u^{(p,s)}$ is a $n(p, s) \times n(p, s)$ -matrix, where $n(p, s)$ is the number of paths from p to s through $c(\mathcal{C})$ (or $c(\mathcal{B})$), so each $u^{(p,s)}$ is a block matrix, indexed as follows

$$u^{(p,s)} = \left(u_{q,r}^{(p,s)} \right)_{q,r},$$

where (q, r) runs over all possible $r \in c(\mathcal{C})$ and $q \in c(\mathcal{B})$, that a path from p to s can go through. Since the vertical inclusions, $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{C} \subseteq \mathcal{D}$, do not have multiple edges, each $u_{q,r}^{(p,s)}$ is a $m \times n$ -matrix, where m is the multiplicity of the edge qs in the inclusion $\mathcal{B} \subseteq \mathcal{D}$ and n is the multiplicity of the edge pr in the inclusion $\mathcal{A} \subseteq \mathcal{C}$.

Let $\xi(\cdot)$ (resp. $\eta(\cdot)$) denote the Perron-Frobenius vector for the graph of $\mathcal{A} \subseteq \mathcal{B}$ (resp. $\mathcal{C} \subseteq \mathcal{D}$). Set

$$w(p, q, r, s) = \sqrt{\frac{\xi(p)\eta(s)}{\xi(q)\eta(r)}}.$$

Define a matrix v by

$$v = \bigoplus_{(q,r)} v^{(q,r)},$$

where $v^{(q,r)}$ is a square matrix, which can be written as a block matrix $v^{(q,r)} = \left(v_{p,s}^{(q,r)} \right)_{p,s}$, with each block given by

$$v_{p,s}^{(q,r)} = w(p, q, r, s) \left(u_{q,r}^{(p,s)} \right)^t, \quad (8.2)$$

where (p, q, r, s) runs through all quadruples in $c(\mathcal{A}) \times c(\mathcal{B}) \times c(\mathcal{C}) \times c(\mathcal{D})$, which can be completed to a cycle $p - r - s - q - p$.

The bi-unitary condition says, that one should be able to choose a unitary u as above, such that v is also unitary.

The possible quadruples (p, r, q, s) are completely determined by the two vertical edges pq and rs . Since E_{10} has 9 edges, there are at most 9×9 blocks in u and v . The diagrams for u and v on the next page show which combinations of (pq, rs) occur. The dots indicates the size of a block, and the frames tell which blocks in u (resp. v) belong to the same direct summand of u (resp. v). The two figures are easily deduced from the given inclusion matrices.

Note how one figure can be obtained from the other by reflecting in the main diagonal.

Diagram for u .

$\begin{smallmatrix} rs \\ pq \end{smallmatrix}$	Aa	Ba	Bb	Bc	Cc	Cd	Dd	De	Ee
Aa		•	•	•					•
Ba	•	••	••	••	•			•	
Bb	•	••	••			•	•		
Bc	•	••		•• ••	• •	•	•		
Cc		•		• •	•• ••	••			
Cd			•	•	••	••			
Dd			•	•			•		
De		•						•	
Ee	•								•

Diagram for v .

$\begin{smallmatrix} rs \\ pq \end{smallmatrix}$	Aa	Ba	Bb	Bc	Cc	Cd	Dd	De	Ee
Aa		•	•	•					•
Ba	•	• •	• •	• •	•			•	
Bb	•	• •	• •			•	•		
Bc	•	• •		•• ••	••	•	•		
Cc		•		••	•• ••	• •			
Cd			•	•	• •	• •			
Dd			•	•			•		
De		•						•	
Ee	•								•

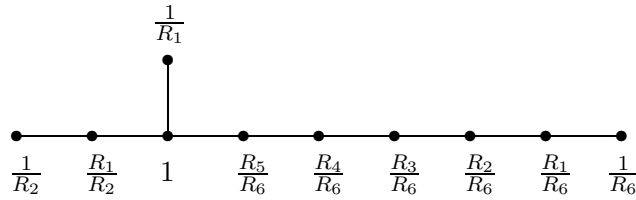
The two figures show that u and v both have 20 direct summands, namely 3 3×3 matrices, 3 2×2 matrices and 14 scalar matrices, and these are subdivided in 36 blocks $u_{q,r}^{(p,s)}$ respectively $v_{q,r}^{(p,s)}$.

As in chapter I we let $R_n(\lambda)$ denote the n 'th degree polynomial, defined inductively by

$$R_0(\lambda) = 1, \quad R_1(\lambda) = \lambda, \quad R_n(\lambda) = \lambda R_{n-1}(\lambda) - R_{n-2}(\lambda).$$

For $\lambda \geq 2$, $R_n(\lambda)$ is positive for all n and increasing in n . For simplicity we will denote $R_n(\lambda(E_{10}))$ by R_n , where $\lambda(E_{10})$ is the largest eigenvalue of the adjacency matrix of E_{10} .

$\lambda(E_{10}) \approx 2.006594$. The corresponding eigenvector (the Perron-Frobenius eigenvector) ξ is given by (see chapter I)



and the eigenvalue equation gives

$$\frac{R_1}{R_2} + \frac{1}{R_1} + \frac{R_5}{R_6} = \lambda(\Gamma) = R_1. \quad (8.3)$$

In the following figure we list the transformation factors

$$w(p, q, r, s) = \sqrt{\frac{\xi(p)\xi(s)}{\xi(q)\xi(r)}}$$

($\xi = \eta$ because the two vertical graphs are equal). Note that $w(p, q, r, s)$ only depends on the two edges pq and rs .

Table of $w(p, q, r, s)^2$									
$\begin{smallmatrix} rs \\ pq \end{smallmatrix}$	Aa	Ba	Bb	Bc	Cc	Cd	Dd	De	Ee
Aa		$\frac{1}{R_2}$	$\frac{1}{R_1^2}$	$\frac{R_5}{R_1 R_6}$					1
Ba	R_2	1	$\frac{R_2}{R_1^2}$	$\frac{R_2 R_5}{R_1 R_6}$	$\frac{R_2 R_5}{R_1 R_4}$			1	
Bb	R_1^2	$\frac{R_1^2}{R_2}$	1			$\frac{R_1 R_3}{R_4}$	$\frac{R_1 R_3}{R_2}$		
Bc	$\frac{R_1 R_6}{R_5}$	$\frac{R_1 R_6}{R_2 R_5}$		1	$\frac{R_6}{R_4}$	$\frac{R_3 R_6}{R_4 R_5}$	$\frac{R_3 R_6}{R_2 R_5}$		
Cc		$\frac{R_1 R_4}{R_2 R_5}$		$\frac{R_4}{R_6}$	1	$\frac{R_3}{R_5}$			
Cd			$\frac{R_4}{R_1 R_3}$	$\frac{R_4 R_5}{R_3 R_6}$	$\frac{R_5}{R_3}$	1			
Dd			$\frac{R_2}{R_1 R_3}$	$\frac{R_2 R_5}{R_3 R_6}$			1		
De		1						1	
Ee	1								1

The squares $\left\|u_{q,r}^{(p,s)}\right\|^2$ of 2–Hilbert–Schmidt norms of the blocks of u , can be determined as follows. Using that the scalar summands in u and v must have modulus 1, together with the transformation formula (8.2), one immediately finds 11 of the 36 norms. The fact that rows and columns in a unitary matrix have 2–norm equal to 1, combined with the transformation formula (8.2), shows that there is at most one possible value for each (pq, rs) . The values are listed in the following diagram

$\begin{array}{c} rs \\ pq \end{array}$		Table of $\left\ u_{q,r}^{(p,s)}\right\ ^2$								
		Aa	Ba	Bb	Bc	Cc	Cd	Dd	De	Ee
Aa			1	1	1					1
Ba		$\frac{1}{R_2}$	$\frac{R_2-1}{R_2}$	1	$\frac{R_7}{R_2R_5}$	$\frac{R_1R_4}{R_2R_5}$			1	
Bb		$\frac{1}{R_1^2}$	$\frac{R_2}{R_1^2}$	1			$\frac{R_4}{R_1R_3}$	$\frac{R_2}{R_1R_3}$		
Bc		$\frac{R_5}{R_1R_6}$	$\frac{R_7}{R_1R_6}$		$1+\frac{R_1R_4}{R_2R_5}$	$\frac{R_7}{R_2R_5}$	$\frac{R_2}{R_1R_3}$	$\frac{R_4}{R_1R_3}$		
Cc			1		$\frac{R_3}{R_5}$	$2-\frac{R_3}{R_5}$	1			
Cd				1	$1-\frac{R_3}{R_5}$	$\frac{R_3}{R_5}$	1			
Dd				1	1			1		
De			1						1	
Ee		1								1

All the numbers listed are clearly positive. To check, that this table of $\|u_{q,r}^{(p,s)}\|^2$ is an admissible solution to the square-norm problem, we have to check:

1. Sums of rows (resp. columns) within each frame should be 1 or 2, according to the number of rows (resp. columns) in the block. (See the diagram of the blocks of u .)
 - (a) $\frac{1}{R_1^2} + \frac{R_2}{R_1^2} = 1$.
 - (b) $\frac{R_5}{R_1 R_6} + \frac{R_7}{R_1 R_6} = 1$.
 - (c) $\frac{1}{R_2} + \frac{1}{R_1^2} + \frac{R_5}{R_1 R_6} = 1$.
 - (d) $\frac{R_2-1}{R_2} + \frac{R_2}{R_1^2} + \frac{R_7}{R_1 R_6} = 2$.
 - (e) $\frac{R_7}{R_2 R_5} + \frac{R_1 R_4}{R_2 R_5} = 1$.
 - (f) $\frac{R_4}{R_1 R_3} + \frac{R_2}{R_1 R_3} = 1$.
2. Moreover, by the transformation formula (8.2)

$$\|v_{p,s}^{(q,r)}\|^2 = w(p, q, r, s)^2 \|u_{q,r}^{(p,s)}\|^2, \quad (8.4)$$

so by multiplying the numbers in the table of $\left\|u_{q,r}^{(p,s)}\right\|^2$, with the numbers in the table of the transition factors, one should obtain an admissible solution to the square-norm problem for v . Because of the properties of the tables on page 95, the solution to the square-norm problem for v , should just be a reflection of the table of $\left\|u_{q,r}^{(p,s)}\right\|^2$ in the main diagonal. Hence we must show that

- (g) $\frac{R_7}{R_2 R_5} \frac{R_6}{R_4} = \frac{R_3}{R_5}$ (position $Bc - Cc$).
- (h) $\frac{R_2}{R_1 R_3} \frac{R_3 R_6}{R_4 R_5} = 1 - \frac{R_3}{R_5}$ (position $Bc - Cd$).
- (i) $\frac{R_4}{R_1 R_3} \frac{R_3 R_6}{R_2 R_5} = 1$ (position $Bc - Dd$).

All the other identities are trivially true.

Since $R_1 = \lambda$, the recursion formula for the R'_n s can be rewritten as

$$R_1 R_n = R_{n-1} + R_{n+1}, \quad n \geq 1.$$

From this, and $R_0 = 1$, it follows by induction on m , that

$$R_m R_n = R_{n-m} + R_{n-m+2} + \cdots + R_{m+n-2} + R_{m+n}, \quad n \geq m. \quad (8.5)$$

This proves (a), (b), (e) and (f). Moreover (c) follows from the equation (8.3) by dividing with R_1 . Also (d) follows by subtracting (c) from the sum of (a) and (b). The remaining equations can be rewritten as

- (g') $R_6 R_7 = R_2 R_3 R_4$.
- (h') $R_2 R_6 = R_1 R_4 (R_5 - R_3)$.
- (i') $R_4 R_6 = R_1 R_2 R_5$.

Multiplying the equation (8.3) by $R_1 R_2 R_6$ yields

$$R_1^2 R_6 + R_2 R_6 + R_1 R_2 R_5 = R_1^2 R_2 R_6,$$

which, by repeated use of (8.5), transforms to

$$R_{10} - R_6 - R_4 = 0.$$

Using (8.5) once more, we get

$$\begin{aligned} R_6 R_7 - R_2 R_3 R_4 &= R_3 (R_{10} - R_6 - R_4) = 0, \\ R_1 R_4 (R_5 - R_3) - R_2 R_6 &= R_{10} - R_6 - R_4 = 0, \\ R_4 R_6 - R_1 R_2 R_5 &= R_{10} - R_6 - R_4 = 0, \end{aligned}$$

which proves (g), (h) and (i).

We can now write up an explicit solution to the bi-unitarity condition.

By (c) $(\sqrt{\frac{1}{R_2}}, \frac{1}{R_1}, \sqrt{\frac{R_5}{R_1 R_6}})$ is a unit vector in \mathbb{R}^3 , so we can find $x_{i,j} \in \mathbb{R}$, $i = 1, 2, 3$, $j = 1, 2$, such that

$$Y = \begin{pmatrix} \sqrt{\frac{1}{R_2}} & x_{11} & x_{12} \\ \frac{1}{R_1} & x_{21} & x_{22} \\ \sqrt{\frac{R_5}{R_1 R_6}} & x_{31} & x_{32} \end{pmatrix}$$

is a unitary matrix.

By (a) and (b)

$$\begin{aligned} x_{11}^2 + x_{12}^2 &= \frac{R_2 - 1}{R_2}, \\ x_{21}^2 + x_{22}^2 &= \frac{R_2}{R_1^2}, \\ x_{31}^2 + x_{32}^2 &= \frac{R_7}{R_1 R_6}. \end{aligned}$$

Hence there exists 3 unit vectors, $e = (e_1, e_2)$, $f = (f_1, f_2)$ and $g = (g_1, g_2)$ in \mathbb{R}^2 , such that

$$Y = \begin{pmatrix} \sqrt{\frac{1}{R_2}} & \sqrt{\frac{R_2 - 1}{R_2}}(e_1, e_2) \\ \frac{1}{R_1} & \sqrt{\frac{R_2}{R_1^2}}(f_1, f_2) \\ \sqrt{\frac{R_5}{R_1 R_6}} & \sqrt{\frac{R_7}{R_1 R_6}}(g_1, g_2) \end{pmatrix}.$$

Moreover, by a change of basis, we may assume that $g = (1, 0)$.

Set

$$\begin{aligned} \sigma_1 &= \sqrt{\frac{1}{R_2}}, & \sigma_2 &= \sqrt{\frac{R_2 - 1}{R_2}}, & \sigma_3 &= \frac{1}{R_1}, & \sigma_4 &= \sqrt{\frac{R_2}{R_1}}, \\ \sigma_5 &= \sqrt{\frac{R_5}{R_1 R_6}}, & \sigma_6 &= \sqrt{\frac{R_7}{R_1 R_6}}, & \tau_1 &= \sqrt{\frac{R_7}{R_2 R_5}}, & \tau_2 &= \sqrt{\frac{R_1 R_4}{R_2 R_5}}, \\ \rho_1 &= \sqrt{\frac{R_4}{R_1 R_3}}, & \rho_2 &= \sqrt{\frac{R_2}{R_1 R_3}}, & \mu_1 &= \sqrt{\frac{R_3}{R_5}}, & \mu_2 &= \sqrt{1 - \frac{R_3}{R_5}}. \end{aligned}$$

Then by (a), (b), (c), (d), (e) and (f)

$$\begin{aligned} \sigma_1^2 + \sigma_2^2 &= 1, & \sigma_3^2 + \sigma_4^2 &= 1, & \sigma_5^2 + \sigma_6^2 &= 1, \\ \sigma_1^2 + \sigma_3^2 + \sigma_5^2 &= 1, & \sigma_2^2 + \sigma_4^2 + \sigma_6^2 &= 2, \\ \tau_1^2 + \tau_2^2 &= 1, & \rho_1^2 + \rho_2^2 &= 1, & \text{and } \mu_1^2 + \mu_2^2 &= 1. \end{aligned}$$

Define $u = \sum_{p,s}^{\oplus} u^{(p,s)}$ by the table in figure 8

Then each of the 20 directed summands in u , indicated by the frames, are unitary by the construction of the unit vectors e and f . Moreover the transformation formula (8.2), together with (g), (h) and (i), show that v is simply the mirror image of figure 8 in the main diagonal (transposing all matrices), and hence v is also unitary. The conclusion is, that there exists a commuting square with the inclusion matrices given in (8.1).

It is elementary to check that

$$\begin{pmatrix} 0 & P(GG^t) \\ P(GG^t) & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & P(G^t G) \\ P(G^t G) & 0 \end{pmatrix}$$

$\begin{array}{c} rs \\ pq \end{array}$	Aa	Ba	Bb	Bc	Cc	Cd	Dd	De	Ee
Aa		1	1	1					1
Ba	σ_1	$\sigma_2(e_1, e_2)$	(f_1, f_2)	$(\tau_1, 0)$	τ_2			1	
Bb	σ_3	$\sigma_4(f_1, f_2)$	$(f_2, -f_1)$			ρ_1	ρ_2		
Bc	σ_5	$(\sigma_6, 0)$		$\begin{pmatrix} \tau_2 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -\tau_1 \\ 0 \end{pmatrix}$	ρ_2	$-\rho_1$		
Cc		1		$\begin{pmatrix} -\mu_1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} \mu_2 & 0 \\ 0 & 1 \end{pmatrix}$	$(1, 0)$			
Cd			1	μ_2	$(\mu_1, 0)$	$(0, 1)$			
Dd			1	-1			1		
De		1						1	
Ee	1								1

Figure 1: The summands of u .

are adjacency matrices for connected bi-partite graphs, so the construction gives a subfactor of the hyperfinite II_1 -factor, with index $\lambda(E_{10})^2$. Furthermore G clearly satisfies Wenzl's criterion for irreducibility of the pair of hyperfinite II_1 -factors (see [Wen2]). Hence we have proved:

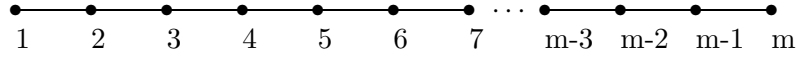
There is an irreducible subfactor of the hyperfinite II_1 -factor of index $\lambda(E_{10})^2 \approx 4.026418$.

By [G.H.J.] chapter 4, this is the lowest value of $\lambda(\Gamma)^2$ above 4, which can be obtained from a finite graph Γ .

Commuting Squares Based on Dynkin Diagrams of Type A

9 Preliminaries

We shall consider the graph A_m



The adjacency matrix for A_m is $H = (h_{i,j})_{i,j=1}^m$ with

$$h_{i,j} = \begin{cases} 1 & \text{if } |i - j| = 1 \\ 0 & \text{otherwise} \end{cases}$$

Define inductively polynomials $R_n(t)$ by

$$\begin{aligned} R_0(t) &= 1 \\ R_1(t) &= t \\ R_{n+1}(t) &= tR_n(t) - R_{n-1}(t). \end{aligned}$$

Note that if $0 \leq t \leq 2$, $t = 2 \cos x$ for some x , we have

$$R_n(t) = \frac{\sin((n+1)x)}{\sin(x)}$$

and if $t > 2$ with $t = 2 \cosh x$ for some x , we have

$$R_n(t) = \frac{\sinh((n+1)x)}{\sinh(x)}.$$

Lemma 9.1 For $l < m$ we denote $R_l(H)$ by $H^{(l)} = (h_{i,j}^{(l)})_{i,j=1}^m$, and we have

1. For l even, $l = 2n$

$$h_{i,j}^{(2n)} = \begin{cases} 1 & \text{if } |i-j| = 0, & n+1 \leq i, j \leq m-n \\ 1 & \text{if } |i-j| = 2, & n \leq i, j \leq m-n+1 \\ 1 & \text{if } |i-j| = 4, & n-1 \leq i, j \leq m-n+2 \\ \vdots & \\ 1 & \text{if } |i-j| = 2n-2, & 2 \leq i, j \leq m-1 \\ 1 & \text{if } |i-j| = 2n, & 1 \leq i, j \leq m \\ 1 & \text{otherwise} \end{cases}$$

or in a more compact notation

$$h_{i,j}^{(2n)} = \begin{cases} 1 & \text{if } |i-j| = 2k, n+1-k \leq i, j \leq m-n+k \\ 0 & \text{otherwise} \end{cases} \quad (9.1)$$

2. For l odd, $l = 2n+1$

$$h_{i,j}^{(2n+1)} = \begin{cases} 1 & \text{if } |i-j| = 1, & n+1 \leq i, j \leq m-n \\ 1 & \text{if } |i-j| = 3, & n \leq i, j \leq m-n+1 \\ 1 & \text{if } |i-j| = 5, & n-1 \leq i, j \leq m-n+2 \\ \vdots & \\ 1 & \text{if } |i-j| = 2n-1, & 2 \leq i, j \leq m-1 \\ 1 & \text{if } |i-j| = 2n+1, & 1 \leq i, j \leq m \\ 1 & \text{otherwise} \end{cases}$$

that is

$$h_{i,j}^{(2n+1)} = \begin{cases} 1 & \text{if } |i-j| = 2k+1, & n+1-k \leq i, j \leq m-n+k \\ 0 & \text{otherwise} \end{cases}$$

Proof The assertion is obviously true for $l = 0$ and $l = 1$. Assume the statement is true for all $l \leq p$, $p \geq 1$.

By definition $H^{(p+1)} = HH^{(p)} - H^{(p-1)}$.

If p is even, $p = 2n$, we have

$$\begin{aligned} (HH^{(2n)})_{i,j} &= \sum_k h_{i,k} h_{k,j}^{(2n)} \\ &= h_{i,i-1} h_{i-1,j}^{(2n)} + h_{i,i+1} h_{i+1,j}^{(2n)} \\ &= h_{i-1,j}^{(2n)} + h_{i+1,j}^{(2n)} \end{aligned}$$

Hence we may view $HH^{(2n)}$ as $K+L$, where $(K)_{i,j} = h_{i-1,j}^{(2n)}$ and $(L)_{i,j} = h_{i+1,j}^{(2n)}$. From the hypothesis on $H^{(2n)}$ we then get

$$(K)_{i,j} = \begin{cases} 1 & \text{if } |i-j| = 2k+1, & n+1-k \leq i, j \leq m-n+k+1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$(L)_{i,j} = \begin{cases} 1 & \text{if } |i-j| = 2k+1, \quad n-k \leq i, j \leq m-n+k \\ 0 & \text{otherwise} \end{cases}$$

and we have

$$h_{i,j}^{(2n+1)} = \begin{cases} 1 & \text{if } |i-j| = 2k+1, \quad n+1-k \leq i, j \leq m-n+k \\ 0 & \text{otherwise} \end{cases}$$

If p is odd, $p = 2n-1$, we have

$$(HH^{(2n-1)})_{i,j} = h_{i-1,j}^{(2n-1)} + h_{i+1,j}^{(2n-1)}$$

and again we can view $HH^{(2n-1)}$ as $K + L$.

This time we have

$$(K)_{i,j} = \begin{cases} 1 & \text{if } |i-j| = 2k, \quad n+1-k \leq i, j \leq m-n+k+1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$(L)_{i,j} = \begin{cases} 1 & \text{if } |i-j| = 2k, \quad n-k \leq i, j \leq m-n+k \\ 0 & \text{otherwise} \end{cases}$$

and since

$$h_{i,j}^{(2n-2)} = \begin{cases} 1 & \text{if } |i-j| = 2k, \quad n-k \leq i, j \leq m-n+k+1 \\ 0 & \text{otherwise} \end{cases}$$

we get the desired result. □

10 The Blocks of the Bi-unitary

We shall determine commuting squares of the form

$$\begin{array}{ccc} C & \subset_{G^t} & D \\ \cup_H & & \cup_F \\ A & \subset_G & B \end{array} \quad (10.1)$$

where G is the adjacency matrix of A_m viewed as a bi-partite graph, and H, F are defined by:

$$\text{If } l \text{ is even } R_l \begin{pmatrix} 0 & G^t \\ G & 0 \end{pmatrix} = \begin{pmatrix} F & 0 \\ 0 & H \end{pmatrix}$$

$$\text{If } l \text{ is odd } R_l \begin{pmatrix} 0 & G^t \\ G & 0 \end{pmatrix} = \begin{pmatrix} 0 & F \\ H & 0 \end{pmatrix}$$

If u is the bi-unitary matrix associated with a commuting square of the form (10.1), then an entry in u is specified by loops of the form



This means that we can describe the entries of u the following way

1. If l is even. For any two edges of A_m (considered bi-partite) e, f there corresponds an entry of u , if there exists edges μ, ν in the graphs corresponding to the matrices F, H such that μ joins the odd labeled vertices of e and f , and ν joins the even labeled vertices of e and f .
2. If l is odd. For any two edges of A_m (considered bi-partite) e, f there corresponds an entry of u , if there exists edges μ, ν in the graphs corresponding to the matrices F, H such that μ joins the odd labeled vertex of e and the even labeled vertex of f , and ν joins the two other vertices of e and f .

Hence we can get a picture of the entries of u in the form of a diagram like

	1	2	3	4	5	6	7	8
1								
2								
3								
4								
5								
6			x					
7								
8								

Where a box in the diagram will be filled if it corresponds to an entry of u . Like the one with “x” in it, will correspond to an entry of u , for l even if the loop



exists, and in the case l odd if the loop



exists.

For the determination of which of the entries in the diagram corresponds to entries of u , we introduce the following notation

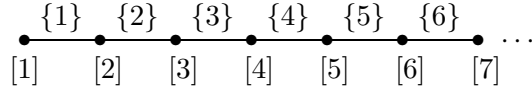
The edge in A_m joining vertex $2j - 1$ to vertex $2j$ is called $\{2j - 1\}$

The edge in A_m joining vertex $2j + 1$ to vertex $2j$ is called $\{2j\}$

We will also denote vertices of A_m by the respective number put in square brackets, e.g. $[j]$, in order to distinguish a vertex from a number.

Finally we denote by α_j the coordinate of the Perron–Frobenius eigenvector of A_m , corresponding to the vertex $[j]$.

Hence the edges and vertices of A_m are labeled by



and we will label a box in the diagram by the corresponding edges of A_m , e.g. the box with an x in it, will be labeled $(\{3\}, \{7\})$.

Lemma 10.1 *For $R_l(A_m)$, $l \leq m$ the boxes in the diagram corresponding to entries of u are given by*

$$(\{1 + t + s\}, \{l - t + s\}) \text{ where } t \in \{0, 1, \dots, l - 1\}, s \in \{0, 1, \dots, m - l - 1\}$$

and

$$(\{1 + t + s\}, \{l + 1 + t - s\}) \text{ where } t \in \{0, 1, \dots, m - l - 2\}, s \in \{0, 1, \dots, l\}$$

Proof First we note the following symmetries. If $(\{j\}, \{k\})$ defines an entry of u , then so do $(\{k\}, \{j\})$, $(\{m - j\}, \{m - k\})$ and $(\{m - k\}, \{m - j\})$.

Consider l even, $l = 2n$.

We first look at $(\{1 + t\}, \{2n - t\})$, $t \in \{0, 1, \dots, 2n - 1\}$.

If t is even, $t = 2s$, we must show that there is an edge between

1. $[2s + 2]$ and $[2n - 2s]$
2. $[2s + 1]$ and $[2n - 2s + 1]$

Ad 1. We may assume $2s+2 \leq 2n-2s$, since the other case will be settled by the above symmetries. From the form of $R_l(A_m)$ we need to look at $|2n-2s-2s-2| = 2n-4s-2$ which corresponds to $k = n-2s-1$ in (9.1). The desired edge exists if

$$\begin{array}{ccccc} n+1-(n-2s-1) & \leq & 2s+2, 2n-2s & \leq & m-n+(n-2s-1) \\ \Updownarrow & & & & \\ 2s+2 & \leq & 2s+2, 2n-2s & \leq & m-2s+1 \end{array}$$

We thus have that $[2s+2]$ is connected to $[2n-2s]$, and that $[2s+2]$ is not connected to a vertex with lower index than $2n-2s$.

Ad 2. Here the corresponding value of k in (9.1) is $n-2s$, and we must have $2s+1 \leq 2s+1, 2n-2s+1 \leq m-2s$. Hence $[2s+1]$ is connected to $[2n-2s+1]$ and not connected to any vertex with lower index than $2n-2s+1$.

For t even we conclude that $(\{1+t\}, \{2n-t\})$ corresponds to an entry of u , and no box to the left of $(\{1+t\}, \{2n-t\})$ defines an entry of u .

If t is odd, $t = 2s+1$, we must show that there is an edge between

1. $[2s+3]$ and $[2n-2s-1]$
2. $[2s+2]$ and $[2n-2s]$

Calculating as before we get

Ad 1. $2s+3 \leq 2s+3, 2n-2s-1 \leq m-2s-2$.

Ad 2. $2s+2 \leq 2s+2, 2n-2s \leq m-2s-1$.

In either case we see that the edge exists, and as above we may conclude that for t odd $(\{1+t\}, \{2n-t\})$ corresponds to an entry of u , and no box to the left of $(\{1+t\}, \{2n-t\})$ defines an entry of u .

By the noted symmetries, we can also conclude that

$$(\{m-2n+t\}, \{m-1-t\}), \quad t \in \{0, 1, \dots, 2n-1\}$$

defines an entries of u , and no box to the right of these defines an entry of u .

We now look at $(\{2n+1+t\}, \{1+t\}), t \in \{0, 1, \dots, m-2n-2\}$.

If t is even, $t = 2s$, we must show that we have edges joining

1. $[2n+2s+1]$ and $[2s+1]$
2. $[2n+2s+2]$ and $[2s+2]$

Since the numerical difference of the indices is $2n$ in both cases, we see from the condition in (9.1) that the box defines an entry of u , and that no box to the right of it does.

If t is odd, $t = 2s + 1$, we must show that we have edges joining

1. $[2n + 2s + 3]$ and $[2s + 3]$
2. $[2n + 2s + 2]$ and $[2s + 2]$

The argument from t even also works here, and we conclude that

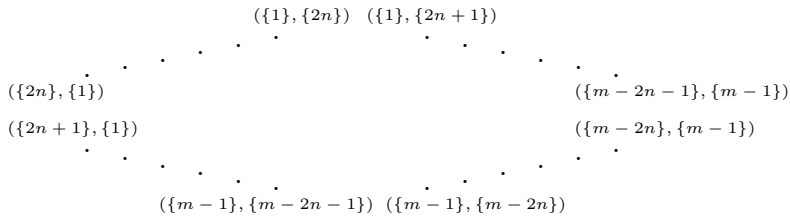
$$(\{2n + 1 + t\}, \{1 + t\}), t \in \{0, 1, \dots, m - 2n - 2\}$$

defines entries of u , and no box to the left of these boxes does define an entry of u . We also get

$$(\{1 + t\}, \{2n + 1 + t\}), t \in \{0, 1, \dots, m - 2n - 2\}$$

defines entries of u , and no box to the right of these boxes does define an entry of u .

I.e. we have now established that the following boxes define entries of u :



The same type of argument, or using the convex nature of the criterion in (9.1), now gives that all edge-pairs in the above sketched diamond defines entries of u .

For odd l we argue similarly. □

Example A_8, R_3

	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{5\}$	$\{6\}$	$\{7\}$
$\{1\}$			x	x			
$\{2\}$		x	x	x	x		
$\{3\}$	x	x	x	x	x	x	
$\{4\}$	x	x	x	x	x	x	x
$\{5\}$		x	x	x	x	x	x
$\{6\}$			x	x	x	x	
$\{7\}$				x	x		

11 The Bi-unitary Condition

We first consider l even.

The boxes of the diagram that may define a block of u are of the form

$$\begin{array}{cc} (\{2j-1\}, \{2k\}) & (\{2j-1\}, \{2k+1\}) \\ (\{2j\}, \{2k\}) & (\{2j\}, \{2k+1\}) \end{array} \quad (11.1)$$

and the ones that may determine a block of v are of the form

$$\begin{array}{cc} (\{2k\}, \{2j-1\}) & (\{2k+1\}, \{2j-1\}) \\ (\{2k\}, \{2j\}) & (\{2k+1\}, \{2j\}) \end{array} \quad (11.2)$$

since these are the labeling of edges which keep the required vertices fixed.

If we by $(\{j\}, \{k\})_u$ and $(\{j\}, \{k\})_v$ denote the value of the entry of u resp. v , corresponding to the edge-pair $(\{j\}, \{k\})$, then the bi-unitary condition is

$$\begin{aligned} (\{2j-1\}, \{2k\})_v &= \sqrt{\frac{\alpha_{2k+1}\alpha_{2j}}{\alpha_{2k}\alpha_{2j-1}}} (\{2j-1\}, \{2k\})_u \\ (\{2j-1\}, \{2k+1\})_v &= \sqrt{\frac{\alpha_{2k+1}\alpha_{2j}}{\alpha_{2k+2}\alpha_{2j-1}}} (\{2j-1\}, \{2k+1\})_u \\ (\{2j\}, \{2k\})_v &= \sqrt{\frac{\alpha_{2k+1}\alpha_{2j}}{\alpha_{2k}\alpha_{2j+1}}} (\{2j\}, \{2k\})_u \\ (\{2j\}, \{2k+1\})_v &= \sqrt{\frac{\alpha_{2k+1}\alpha_{2j}}{\alpha_{2k+2}\alpha_{2j+1}}} (\{2j\}, \{2k+1\})_u \end{aligned} \quad (11.3)$$

If we can find a solution with

$$\begin{aligned}
(\{2j-1\}, \{2k\})_u &= \sqrt{\frac{\alpha_{2k+1}\alpha_{2j}}{\alpha_{2k}\alpha_{2j-1}}}(\{2k\}, \{2j-1\})_u \\
(\{2j-1\}, \{2k+1\})_u &= \sqrt{\frac{\alpha_{2k+1}\alpha_{2j}}{\alpha_{2k+2}\alpha_{2j-1}}}(\{2k+1\}, \{2j-1\})_u \\
(\{2j\}, \{2k\})_u &= \sqrt{\frac{\alpha_{2k+1}\alpha_{2j}}{\alpha_{2k}\alpha_{2j+1}}}(\{2k\}, \{2j\})_u \\
(\{2j\}, \{2k+1\})_u &= \sqrt{\frac{\alpha_{2k+1}\alpha_{2j}}{\alpha_{2k+2}\alpha_{2j+1}}}(\{2k+1\}, \{2j\})_u
\end{aligned}$$

we will get

$$\begin{aligned}
(\{2j-1\}, \{2k\})_u &= (\{2k\}, \{2j-1\})_v = x_{1,1} \\
(\{2j-1\}, \{2k+1\})_u &= (\{2k+1\}, \{2j-1\})_v = x_{1,2} \\
(\{2j\}, \{2k\})_u &= (\{2k\}, \{2j\})_v = x_{2,1} \\
(\{2j\}, \{2k+1\})_u &= (\{2k+1\}, \{2j\})_v = x_{2,2}
\end{aligned}$$

so if the block $\begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix}$ of u is unitary, then a corresponding block of v also equals $\begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix}$, and hence v becomes a unitary matrix.

For l odd the situation is different.

The boxes of the diagram that may define a block of u are of the form

$$\begin{array}{cc}
(\{2j-1\}, \{2k-1\}) & (\{2j-1\}, \{2k\}) \\
(\{2j\}, \{2k-1\}) & (\{2j\}, \{2k\})
\end{array} \tag{11.4}$$

and those that may define a block of v are of the form

$$\begin{array}{cc}
(\{2j\}, \{2k\}) & (\{2j\}, \{2k+1\}) \\
(\{2j+1\}, \{2k\}) & (\{2j+1\}, \{2k+1\})
\end{array} \tag{11.5}$$

and the bi-unitarity condition is

$$\begin{aligned}
(\{2j\}, \{2k\})_v &= \sqrt{\frac{\alpha_{2k}\alpha_{2j}}{\alpha_{2k+1}\alpha_{2j+1}}}(\{2j\}, \{2k\})_u \\
(\{2j\}, \{2k+1\})_v &= \sqrt{\frac{\alpha_{2k+2}\alpha_{2j}}{\alpha_{2k+1}\alpha_{2j+1}}}(\{2j\}, \{2k+1\})_u \\
(\{2j+1\}, \{2k\})_v &= \sqrt{\frac{\alpha_{2k}\alpha_{2j+2}}{\alpha_{2k+1}\alpha_{2j+1}}}(\{2j+1\}, \{2k\})_u \\
(\{2j+1\}, \{2k+1\})_v &= \sqrt{\frac{\alpha_{2k+2}\alpha_{2j+2}}{\alpha_{2k+1}\alpha_{2j+1}}}(\{2j+1\}, \{2k+1\})_u
\end{aligned}$$

12 A Solution

We will first give patterns of signs on the blocks of u and v , which will show that if there is a solution to the problem, then a real solution exists. With this pattern of signs we will then proceed to show which modulus is to be put in each of the boxes determined in lemma 10.1.

Note that if $a, b \in \mathbb{R}$ with $a^2 + b^2 = 1$ then any matrix of the form $\begin{pmatrix} \sigma_1 a & \sigma_2 b \\ \sigma_3 b & \sigma_4 a \end{pmatrix}$, with σ_i denoting either $+$ or $-$ will be unitary if exactly three of the σ_i 's are equal.

We will assign a sign to each box in the diagram, such that any of the u and v patterns described in lemma 10.1 will correspond to unitary matrices, provided that the a 's and b 's we assign to the entries, are such that $a^2 + b^2 = 1$.

For l even the blocks of u and v coincide, and the extra condition we have put on our desired solution, implies that any pattern of sign must be symmetric with respect to the main diagonal. A pattern that will do the job is found in figure 2.

For l odd a pattern is found in figure 3.

Theorem 12.1 *With the signs listed previously, the following is a solution of the bi-unitary matrix u .*

1. For $t \in \{0, 1, \dots, l-1\}$ and $s \in \{0, 1, \dots, m-l-1\}$ put

$$(\{1+s+t\}, \{l-t+s\})_u = \sqrt{\frac{\alpha_{1+t}\alpha_{l-t}}{x_1 x_2}}$$

where

$$x_1 = \begin{cases} \alpha_{2+s+t} & \text{if } s+t \text{ is even} \\ \alpha_{1+s+t} & \text{if } s+t \text{ is odd} \end{cases}$$

$$x_2 = \begin{cases} \alpha_{l-t+s+1} & \text{if } s-t \text{ is even} \\ \alpha_{l-t+s} & \text{if } s-t \text{ is odd} \end{cases}$$

2. For $t \in \{0, 1, \dots, m-l-2\}$ and $s \in \{0, 1, \dots, l\}$ put

$$(\{1+s+t\}, \{l+t-s+1\})_u = \sqrt{\frac{\alpha_{1+t}\alpha_{l+2+t}}{y_1 y_2}}$$

where

$$y_1 = \begin{cases} \alpha_{2+s+t} & \text{if } s+t \text{ is even} \\ \alpha_{1+s+t} & \text{if } s+t \text{ is odd} \end{cases}$$

$$y_2 = \begin{cases} \alpha_{l+t-s+1} & \text{if } s-t \text{ is even} \\ \alpha_{l+t-s+2} & \text{if } s-t \text{ is odd} \end{cases}$$

Proof We will first disregard the limitations on which boxes of the diagram that correspond to entries of u . We will do this by assuming that every box gets a number assigned to it via the

statement of the theorem, and by putting $\alpha_0 = 0$, and $\alpha_{-n} = -\alpha_n$. This identification of negative labeled α 's is justified since $\alpha_n = \frac{\sin(nx)}{\sin(x)}$ for some x .

In the 2×2 blocks we must check that the sum of the moduli squared in a row or a column equals 1. And for the 1×1 blocks we must check that the assigned scalar is 1.

For $l = 2n$ we have the following block indices from (11.1)

$$\begin{pmatrix} (\{2j-1\}, \{2k\})_u & (\{2j-1\}, \{2k+1\})_u \\ (\{2j\}, \{2k\})_u & (\{2j\}, \{2k+1\})_u \end{pmatrix}$$

Obviously $(\{2j-1\}, \{2k\})_u = (\{2j\}, \{2k+1\})_u$ and $(\{2j-1\}, \{2k+1\})_u = (\{2j\}, \{2k\})_u$. We have that $(\{2j-1\}, \{2k\}) = (\{1+t+s\}, \{2n-t+s\})$ for some choice of s and t . Hence $1+t+s-2n+t-s = 2j-1-2k$ and we must have $t = n+j-k-1$. This gives that the numerator of $(\{2j-1\}, \{2k\})_u$ is

$$\sqrt{\alpha_{n+j-k}\alpha_{n-j+k+1}}$$

We also have that $(\{2j-1\}, \{2k+1\}) = (\{1+t+s\}, \{2n+1+t-s\})$ for some choice of s and t . Hence $1+t+s+2n+1+t-s = 2j+2k$ and we must have $t = j+k-n-1$. This gives that the numerator of $(\{2j-1\}, \{2k+1\})_u$ is

$$\sqrt{\alpha_{j+k-n}\alpha_{n+j+k+1}}$$

In either case the denominator is

$$\sqrt{\alpha_{2j}\alpha_{2k+1}}$$

and hence we must check that

$$\alpha_{n+j-k}\alpha_{n-j+k+1} + \alpha_{j+k-n}\alpha_{n+j+k+1} = \alpha_{2j}\alpha_{2k+1} \quad (12.1)$$

If we put $p = j-k$ and $q = j+k$, (12.1) transforms to

$$\alpha_{n+p}\alpha_{n-p+1} + \alpha_{q-n}\alpha_{n+q+1} = \alpha_{p+q}\alpha_{q-p+1}$$

Since $\alpha_j = \frac{\sin(jx)}{\sin(x)}$ for some x , it is enough to verify

$$\begin{aligned} & (e^{i(n+p)x} - e^{-i(n+p)x})(e^{i(n-p+1)x} - e^{-i(n-p+1)x}) \\ & + (e^{i(q-n)x} - e^{-i(q-n)x})(e^{i(q+n+1)x} - e^{-i(q+n+1)x}) \\ & = (e^{i(p+q)x} - e^{-i(p+q)x})(e^{i(q-p+1)x} - e^{-i(q-p+1)x}) \end{aligned}$$

which is a trivial calculation.

For l odd, $l = 2n+1$, the 2×2 blocks of u are of the form

$$\begin{pmatrix} (\{2j-1\}, \{2k-1\})_u & (\{2j-1\}, \{2k\})_u \\ (\{2j\}, \{2k-1\})_u & (\{2j\}, \{2k\})_u \end{pmatrix}$$

Also here we have $(\{2j-1\}, \{2k-1\})_u = (\{2j\}, \{2k\})_u$ and $(\{2j-1\}, \{2k\})_u = (\{2j\}, \{2k-1\})_u$.

$(\{2j-1\}, \{2k-1\}) = (\{1+s+t\}, \{2n+1-t+s\})$ for some t and s . As before we get that $t = n+k-j$, and that the numerator of $(\{2j-1\}, \{2k-1\})_u$ is

$$\sqrt{\alpha_{1-j+k+n}\alpha_{n+1+j-k}}.$$

Also $(\{2j-1\}, \{2k\}) = (\{1+s+t\}, \{2n+1+1+t-s\})$ for some t and s . We get $t = j+k-n-2$, and that the numerator of $(\{2j-1\}, \{2k\})_u$ is

$$\sqrt{\alpha_{j+k-n-1} \alpha_{n+j+k+1}}.$$

In both cases the denominator is

$$\sqrt{\alpha_{2j} \alpha_{2k}},$$

so we need to verify

$$\alpha_{1+n-p} \alpha_{n+p+1} + \alpha_{q-n-1} \alpha_{n+q+1} = \alpha_{p+q} \alpha_{q-p}$$

where $p = j - k$ and $q = j + k$.

Using exponentials this verification is also easy.

A 2×2 block of v looks like (11.5). By the just determined

$$\begin{aligned} (\{2j\}, \{2k\})_u &= \sqrt{\frac{\alpha_{1-j+k+n} \alpha_{n+1+j-k}}{\alpha_{2j} \alpha_{2k}}} \\ (\{2j\}, \{2k+1\})_u &= \sqrt{\frac{\alpha_{j+k-n} \alpha_{n+j+k+2}}{\alpha_{2j} \alpha_{2k+2}}} \\ (\{2j+1\}, \{2k\})_u &= \sqrt{\frac{\alpha_{j+k-n} \alpha_{n+k+j+2}}{\alpha_{2j+2} \alpha_{2k}}} \\ (\{2j+1\}, \{2k+1\})_u &= \sqrt{\frac{\alpha_{1-j+k+n} \alpha_{n+1+j-k}}{\alpha_{2j+2} \alpha_{2k+2}}} \end{aligned}$$

and from the bi-unitary condition (11.3), we get

$$\begin{aligned} (\{2j\}, \{2k\})_v &= (\{2j+1\}, \{2k+1\})_v = \sqrt{\frac{\alpha_{1-j+k+n} \alpha_{n+1+j-k}}{\alpha_{2j+1} \alpha_{2k+1}}} \\ (\{2j\}, \{2k+1\})_v &= (\{2j+1\}, \{2k\})_v = \sqrt{\frac{\alpha_{j+k-n} \alpha_{n+j+k+2}}{\alpha_{2j+1} \alpha_{2k+1}}} \end{aligned}$$

We thus need to verify

$$\alpha_{1-j+k+n} \alpha_{n+1+j-k} + \alpha_{j+k-n} \alpha_{n+j+k+2} = \alpha_{2j+1} \alpha_{2k+1}$$

which again is easy.

We will now look at the blocks of u and v which are only 1×1 -blocks. The labels of these blocks must be found in the boundary of the diamond determined in lemma 10.1, i.e. among

1. $(\{1+t\}, \{l-t\})$.
2. $(\{m+t-l\}, \{m-t-1\})$.
3. $(\{1+t\}, \{l+t+1\})$.

4. $(\{l + t + 1\}, \{1 + t\})$.

If a block is of the form $\begin{pmatrix} (1,1) & (1,2) \\ (2,1) & (2,2) \end{pmatrix}$ with at least one index defining an entry of u resp. v , then the block does not define a 2×2 -block of u or v , if in the above cases

1. $(\{1 + t\}, \{l - t\})$ defines index $(2, 2)$.
2. $(\{m + t - l\}, \{m - t - 1\})$ defines index $(1, 1)$.
3. $(\{1 + t\}, \{l + t + 1\})$ defines index $(2, 1)$.
4. $(\{l + t + 1\}, \{1 + t\})$ defines index $(1, 2)$.

For l even, $l = 2n$, the blocks of u and of v are determined in (11.1).

Case 1: For t even the index determined is of the form $(\{2j - 1\}, \{2k\})$ and hence part of a 2×2 -block.

For t odd the index determined is of the form $(\{2j\}, \{2k + 1\})$ for $j = \frac{1+t}{2}$ and $k = \frac{2n-t-1}{2}$. The previous calculations now give that the modulus of the corresponding entry is

$$\sqrt{\frac{\alpha_{1+t} \alpha_{2n-t}}{\alpha_{1+t} \alpha_{2n-t}}} = 1.$$

Case 2: For $m + t$ even the index determined is of the form $(\{2j\}, \{2k + 1\})$ and hence part of a 2×2 -block.

For $m + t$ odd the index determined is of the form $(\{2j - 1\}, \{2k\})$ for $j = \frac{m+t-2n+1}{2}$ and $k = \frac{m-t-1}{2}$. And we get that the modulus of the corresponding entry is

$$\sqrt{\frac{\alpha_{1+t} \alpha_{2n-t}}{\alpha_{m+1-2n+t} \alpha_{m-t}}} = 1$$

since $\alpha_{m+1-t} = \alpha_t$.

Case 3: For t even the index determined is of the form $(\{2j - 1\}, \{2k + 1\})$ and hence part of a 2×2 -block.

For t odd the index determined is of the form $(\{2j\}, \{2k\})$ for $j = \frac{t+1}{2}$ and $k = \frac{2n+t+1}{2}$. And we get that the modulus of the corresponding entry is

$$\sqrt{\frac{\alpha_{1+t} \alpha_{2n+t+2}}{\alpha_{1+t} \alpha_{2n+t+2}}} = 1.$$

Case 4: This is settled like case 3.

For l odd, $l = 2n + 1$, the blocks of u and v are given by (11.4) and (11.5).

The 1×1 blocks of u :

Case 1: For t even the index determined is of the form $(\{2j + 1\}, \{2k - 1\})$ and hence part of a 2×2 -block.

For t odd the index determined is of the form $(\{2j\}, \{2k\})$ for $j = \frac{t+1}{2}$ and $k = \frac{2n+1-t}{2}$. And we get that the modulus of the corresponding entry is

$$\sqrt{\frac{\alpha_{1+t} \alpha_{2n+1-t}}{\alpha_{1+t} \alpha_{2n+1-t}}} = 1.$$

Case 2: For $m + t$ odd the index determined is of the form $(\{2j\}, \{2k\})$ and hence part of a 2×2 -block.

For $m + t$ even the index determined is of the form $(\{2j - 1\}, \{2k - 1\})$ for $j = \frac{m+t-2n}{2}$ and $k = \frac{m-t}{2}$. And we get that the modulus of the corresponding entry is

$$\sqrt{\frac{\alpha_{2n+1-t} \alpha_{t+1}}{\alpha_{m+t-2n} \alpha_{m-t}}} = 1.$$

Case 3: For t even the index determined is of the form $(\{2 - j\}, \{2k\})$ and hence part of a 2×2 -block.

For t odd the index determined is of the form $(\{2j\}, \{2k - 1\})$ for $j = \frac{t+1}{2}$ and $k = \frac{2n+t+3}{2}$. And we get that the modulus of the corresponding entry is

$$\sqrt{\frac{\alpha_{t+1} \alpha_{2n+t+3}}{\alpha_{t+1} \alpha_{2n+t+3}}} = 1.$$

Case 4: Is settled like case 3.

The 1×1 blocks of v :

Case 1: For t odd the index determined is of the form $(\{2j\}, \{2k\})$ and hence part of a 2×2 -block.

For t even the index determined is of the form $(\{2j + 1\}, \{2k + 1\})$ for $j = \frac{t}{2}$ and $k = \frac{2n-t}{2}$. And we get that the modulus of the corresponding entry is

$$\sqrt{\frac{\alpha_{2n+1-t} \alpha_{t+1}}{\alpha_{2n+1-t} \alpha_{t+1}}} = 1.$$

Case 2: For $m + t$ even the index determined is of the form $(\{2j + 1\}, \{2k + 1\})$ and hence part of a 2×2 -block.

For $m + t$ odd the index determined is of the form $(\{2j\}, \{2k\})$ for $j = \frac{m+t-2n-1}{2}$ and $k = \frac{m-t-1}{2}$. And we get that the modulus of the corresponding entry is

$$\sqrt{\frac{\alpha_{2n+1-t} \alpha_{t+1}}{\alpha_{m+t-2n} \alpha_{m-t}}} = 1.$$

Case 3: For t odd the index determined is of the form $(\{2j\}, \{2k+1\})$ and hence part of a 2×2 -block.

For t even the index determined is of the form $(\{2j+1\}, \{2k\})$ for $j = \frac{t}{2}$ and $k = \frac{2n+2+t}{2}$. And we get that the modulus of the corresponding entry is

$$\sqrt{\frac{\alpha_{t+1} \alpha_{2n+t+3}}{\alpha_{t+1} \alpha_{2n+t+3}}} = 1.$$

Case 4: Is settled like case 3. □

The calculations in the above proof gives the following

Corollary 12.2 *For l even the moduli of the entries of u are given by*

$$\begin{aligned} (\{2j-1\}, \{2k\})_u &= (\{2j\}, \{2k+1\})_u = \sqrt{\frac{\alpha_{n+j-k} \alpha_{n-j+k+1}}{\alpha_{2j} \alpha_{2k+1}}} \\ (\{2j\}, \{2k\})_u &= (\{2j-1\}, \{2k+1\})_u = \sqrt{\frac{\alpha_{j+k-n} \alpha_{n+j+k+1}}{\alpha_{2j} \alpha_{2k+1}}} \end{aligned}$$

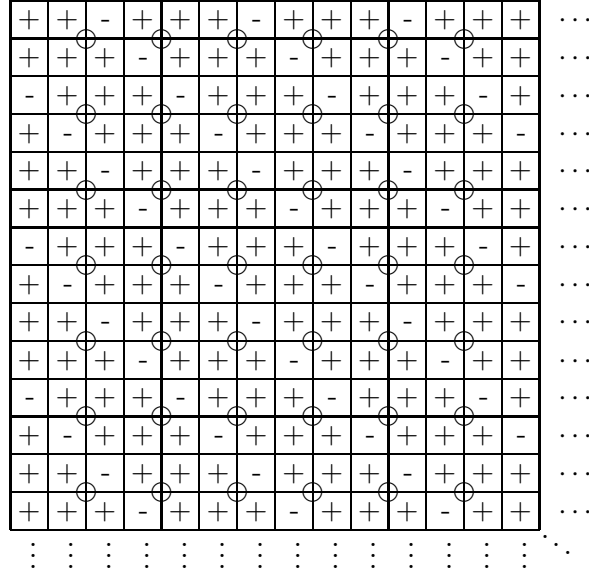
For l odd the moduli of the entries of u are given by

$$\begin{aligned} (\{2j\}, \{2k\})_u &= (\{2j-1\}, \{2k-1\})_u = \sqrt{\frac{\alpha_{1-j+k+n} \alpha_{n+j-k+1}}{\alpha_{2j} \alpha_{2k}}} \\ (\{2j-1\}, \{2k\})_u &= (\{2j\}, \{2k-1\})_u = \sqrt{\frac{\alpha_{j+k-n-1} \alpha_{n+j+k+1}}{\alpha_{2j} \alpha_{2k}}} \end{aligned}$$

Example A_{13}, R_6 . For simplicity we put $\beta_j = \sqrt{\alpha_j}$

$$\begin{array}{cccccccccccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \frac{\beta_6 \beta_1}{\beta_2 \beta_7} & -\frac{\beta_8 \beta_1}{\beta_2 \beta_7} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \frac{\beta_8 \beta_1}{\beta_2 \beta_7} & \frac{\beta_6 \beta_1}{\beta_2 \beta_7} & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \frac{\beta_3 \beta_4}{\beta_4 \beta_5} & -\frac{\beta_8 \beta_1}{\beta_4 \beta_5} & \frac{\beta_2 \beta_5}{\beta_4 \beta_7} & \frac{\beta_2 \beta_9}{\beta_4 \beta_7} & \frac{\beta_6 \beta_1}{\beta_4 \beta_9} & -\frac{\beta_{10} \beta_3}{\beta_4 \beta_9} & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \frac{\beta_8 \beta_1}{\beta_4 \beta_5} & \frac{\beta_3 \beta_4}{\beta_4 \beta_5} & -\frac{\beta_2 \beta_9}{\beta_4 \beta_7} & \frac{\beta_2 \beta_5}{\beta_4 \beta_7} & \frac{\beta_{10} \beta_3}{\beta_4 \beta_9} & \frac{\beta_6 \beta_1}{\beta_4 \beta_9} & -1 & \cdot & \cdot & \cdot \\
\cdot & \frac{\beta_2 \beta_5}{\beta_3 \beta_6} & -\frac{\beta_8 \beta_1}{\beta_3 \beta_6} & \frac{\beta_3 \beta_4}{\beta_5 \beta_6} & \frac{\beta_2 \beta_9}{\beta_5 \beta_6} & \frac{\beta_3 \beta_4}{\beta_7 \beta_6} & -\frac{\beta_{10} \beta_3}{\beta_7 \beta_6} & \frac{\beta_2 \beta_5}{\beta_9 \beta_6} & \frac{\beta_{11} \beta_4}{\beta_9 \beta_6} & \frac{\beta_6 \beta_1}{\beta_{11} \beta_6} & -\frac{\beta_{12} \beta_5}{\beta_{11} \beta_6} & \cdot & \cdot \\
1 & \frac{\beta_8 \beta_1}{\beta_3 \beta_6} & \frac{\beta_2 \beta_5}{\beta_3 \beta_6} & -\frac{\beta_2 \beta_9}{\beta_5 \beta_6} & \frac{\beta_3 \beta_4}{\beta_5 \beta_6} & \frac{\beta_{10} \beta_3}{\beta_7 \beta_6} & \frac{\beta_3 \beta_4}{\beta_7 \beta_6} & -\frac{\beta_{11} \beta_4}{\beta_9 \beta_6} & \frac{\beta_2 \beta_5}{\beta_9 \beta_6} & \frac{\beta_{12} \beta_5}{\beta_{11} \beta_6} & \frac{\beta_6 \beta_1}{\beta_{11} \beta_6} & -1 & \cdot \\
-1 & \frac{\beta_6 \beta_1}{\beta_3 \beta_8} & \frac{\beta_2 \beta_9}{\beta_3 \beta_8} & \frac{\beta_2 \beta_5}{\beta_5 \beta_8} & -\frac{\beta_{10} \beta_3}{\beta_5 \beta_8} & \frac{\beta_3 \beta_4}{\beta_7 \beta_8} & \frac{\beta_{11} \beta_4}{\beta_7 \beta_8} & \frac{\beta_3 \beta_4}{\beta_8 \beta_9} & -\frac{\beta_{12} \beta_5}{\beta_8 \beta_9} & \frac{\beta_2 \beta_5}{\beta_{11} \beta_8} & \frac{\beta_6 \beta_{13}}{\beta_{11} \beta_8} & 1 & \cdot \\
\cdot & -\frac{\beta_2 \beta_9}{\beta_3 \beta_8} & \frac{\beta_6 \beta_1}{\beta_3 \beta_8} & \frac{\beta_{10} \beta_3}{\beta_5 \beta_8} & \frac{\beta_2 \beta_5}{\beta_5 \beta_8} & -\frac{\beta_{11} \beta_4}{\beta_7 \beta_8} & \frac{\beta_3 \beta_4}{\beta_7 \beta_8} & \frac{\beta_{12} \beta_5}{\beta_8 \beta_9} & \frac{\beta_3 \beta_4}{\beta_8 \beta_9} & -\frac{\beta_6 \beta_{13}}{\beta_{11} \beta_8} & \frac{\beta_2 \beta_5}{\beta_{11} \beta_8} & \cdot & \cdot \\
\cdot & \cdot & -1 & \frac{\beta_6 \beta_1}{\beta_5 \beta_{10}} & \frac{\beta_{11} \beta_4}{\beta_5 \beta_{10}} & \frac{\beta_2 \beta_5}{\beta_7 \beta_{10}} & -\frac{\beta_{12} \beta_5}{\beta_7 \beta_{10}} & \frac{\beta_3 \beta_4}{\beta_9 \beta_{10}} & \frac{\beta_6 \beta_{13}}{\beta_9 \beta_{10}} & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & -\frac{\beta_{11} \beta_4}{\beta_5 \beta_{10}} & \frac{\beta_6 \beta_1}{\beta_5 \beta_{10}} & \frac{\beta_{12} \beta_5}{\beta_7 \beta_{10}} & \frac{\beta_2 \beta_5}{\beta_7 \beta_{10}} & -\frac{\beta_6 \beta_{13}}{\beta_9 \beta_{10}} & \frac{\beta_3 \beta_4}{\beta_9 \beta_{10}} & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & -1 & \frac{\beta_6 \beta_1}{\beta_7 \beta_{12}} & \frac{\beta_6 \beta_{13}}{\beta_7 \beta_{12}} & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & -\frac{\beta_6 \beta_{13}}{\beta_7 \beta_{12}} & \frac{\beta_6 \beta_1}{\beta_7 \beta_{12}} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

Figure 2: Sign pattern for l even



Where \bigcirc denotes that the 4 adjacent boxes may span a block of u resp. v

Part II

Infinite Dimensional Commuting Squares

Hyperfiniteness of II_1 -factors From Infinite Dimensional Multi-Matrix Algebras

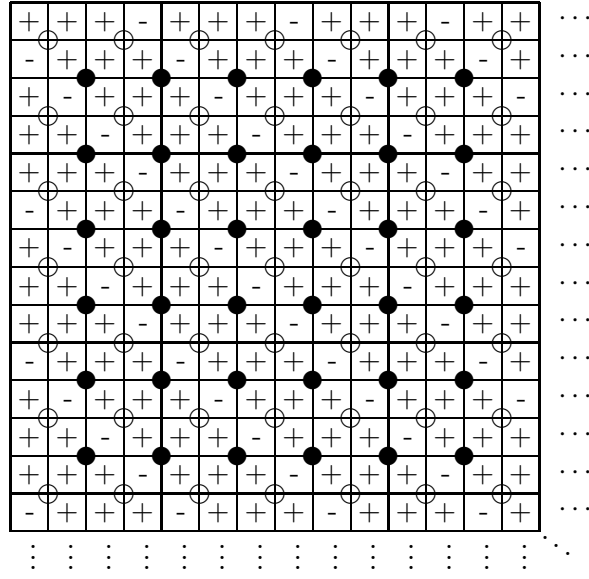
In this chapter we will prove, that the construction of inclusions of Hyperfinite II_1 -factors, based on ladders of multi-matrix algebras, used previously, generalizes to constructions based on, what we have chosen to call infinite dimensional multi-matrix algebras.

13 Preliminaries

An infinite dimensional multi-matrix algebra is an infinite direct sum of full matrix algebras.

$$A = \bigoplus_{j=1}^{\infty} A_j, A_j \cong M_{a_j}(\mathbb{C})$$

Figure 3: Sign pattern for l odd



Where ○ denotes that the 4 adjacent boxes may span a block of u and
 where ● denotes that the 4 adjacent boxes may span a block of v

The *dimension vector* of A is $(a_j)_{j=1}^\infty$

A trace on A is given by its action on each direct summand, i.e. a trace on A is determined by the *trace vector*

$$\alpha = (\alpha_j)_{j=1}^\infty$$

where α_j is the trace of a minimal projection in A_j

The trace tr defined by α is *finite* if

$$tr(\mathbf{1}) = \sum_{j=1}^\infty \alpha_j a_j < \infty \quad (13.1)$$

Note that 13.1 implies that $\|\alpha\|_1 < \infty$, since $a_j \geq 1$

If A, B are infinite dimensional multi-matrix algebras $A \subset B$ we define the *inclusion matrix* G of $A \subset B$ by

$$G = (g_{ij})_{i,j=1}^\infty, g_{ij} = \text{multiplicity of } A_i \text{ in } B_j$$

and we write $A \subset_G B$. If α, β denote trace vectors for A, B defining finite traces which extend one another, and a, b denote the dimension vectors then

$$\alpha = G\beta \text{ and } b = G^t a.$$

All traces on multi-matrix algebras in the following, are assumed to be finite.

In [S] Chapter 6 there is a discussion of countable non-negative matrices $T = (t_{ij})_{i,j=1}^\infty$ under the assumptions

1. $T^k = (t_{ij}^k)_{i,j=1}^\infty$ are all element wise finite.
2. T is irreducible, in the usual Perron-Frobenius-Theory sense.

Theorem 6.4 of [S] states

If $x = (x_i)_{i=1}^\infty$ is a positive right eigenvector of T and $y = (y_i)_{i=1}^\infty$ is a positive left eigenvector of T , both corresponding to the same eigenvalue then (1) $\sum_{i=1}^\infty x_i y_i < \infty$ if and only if (2) x resp. y are multiples of unique right resp. left eigenvectors of T corresponding to the largest eigenvalue of T .

For our purposes we will be interested in infinite, locally finite, connected graphs. If G is the adjacency matrix of a infinite, locally finite, connected graph, then G is symmetric. Hence any left eigenvector of G will also be a right eigenvector. And the above theorem allows us to conclude

Corollary 13.1 *If x is a positive eigenvector of G then*

$\|x\|_2 < \infty \Leftrightarrow x$ *is proportional to the unique positive eigenvector corresponding to the largest eigenvalue of G .*

For our constructions we will actually be interested in positive eigenvectors x satisfying $\|x\|_1 < \infty$, but the above corollary also applies to such vectors.

All inclusion matrices in this chapter will be adjacency matrices for locally finite graphs, which may be finite or countably infinite.

14 Towers of Infinite Multi-Matrix Algebras

Lemma 14.1 *Let $\begin{array}{ccc} B_0 & \subset & B_1 \\ \cup & & \cup \\ A_0 & \subset & A_1 \end{array}$ be a commuting square of infinite dimensional multi-matrix algebras with respect to a trace tr_{B_1} on B_1 .*

Put $B_2 = \langle B_1, e_{B_0} \rangle$, and $A_2 = \{A_1, e_{B_0}\}''$, then

1. $\overline{A_1 e_{B_0} A_1}^{weak} = w A_2$, where $w = \mathcal{Z}_{A_2}(e_{B_0})$, the central support of e_{B_0} in A_2 .
2. If the representation of A_1 on $L^2(A_1, tr_{A_1})$ is denoted by π , we define

$$\phi : A_2 \rightarrow \langle A_1, e_{A_0} \rangle$$

by

$$\phi(x) = \pi(x|_{L^2(A_1, tr_{A_1})}), x \in A_2. \quad (14.1)$$

If tr_{B_1} is a Markov trace of modulus β for $B_0 \subset B_1$, and the restriction to A_1 , tr_{A_1} , is a Markov trace of modulus γ for $A_0 \subset A_1$, then

$$A_2 = z A_2 \oplus (\mathbf{1} - z) A_2,$$

where $z A_2 \cong \langle A_1, e_{A_0} \rangle$ and $(\mathbf{1} - z) A_2 \cong$ a subalgebra of A_1 .

Furthermore $tr_{A_2}(z) = \frac{\gamma}{\beta}$

Proof $A_1 e_{B_0} A_1$ is a $*$ -algebra, and for $a \in A_1$ we have

$$e_{B_0} a e_{B_0} = E_{B_0}(a) e_{B_0} = E_{B_0} E_{A_1}(a) e_{B_0} = E_{A_0}(a) e_{B_0},$$

and

$$e_{B_0} a e_{B_0} = e_{B_0} E_{A_0}(a).$$

Hence $A_1 e_{B_0} A_1$ is a two-sided ideal in $\text{alg}(A_1, e_{B_0})$, and we have that

$\overline{A_1 e_{B_0} A_1}^{weak}$ is a two-sided ideal in $\{A_1, e_{B_0}\}'' = A_2$. In particular there exists a projection $w \in A_2$ such that $\overline{A_1 e_{B_0} A_1}^{weak} = w A_2$, and since $e_{B_0} \in w A_2$ we must have $w \geq \mathcal{Z}_{A_2}(e_{B_0})$, the central

support of e_{B_0} in A_2 . Put $z = \mathcal{Z}_{A_2}(e_{B_0})$. For $x, y \in A_1$ we then have $xe_{B_0}y \in zA_2$, since $z \in A'_2 \subset A'_1$ i.e. $\overline{A_1 e_{B_0} A_1}^{weak} \subset zA_2$, and hence $w \leq z$. This proves $z = w$.

For $a \in A_1$ we have

$$\phi(a) = \pi(a|_{L^2(A_1, tr_{A_1})}) = \pi(a).$$

Since e_{B_0} “lives” on $L^2(B_1, tr_{B_1})$ the restriction to $L^2(A_1) \subset L^2(B_1)$ is just composition with the orthogonal projection from $L^2(B_1)$ to $L^2(A_1)$, and by the commuting square condition this equals e_{A_0} . I.e.

$$\phi(e_{B_0}) = \pi(e_{B_0}|_{L^2(A_1, tr_{A_1})}) = e_{A_0}.$$

Since $\overline{A_1 e_{A_0} A_1}^{weak} = \langle A_1, e_{A_0} \rangle$ we have that

$$\phi(A_1 e_{B_0} A_1) \text{ is dense in } \langle A_1, e_{A_0} \rangle,$$

and since $A_1 e_{B_0} A_1 \subset zA_2$, we get $\phi(zA_2)$ is dense in $\langle A_1, e_{A_0} \rangle$. Let tr_{B_2} be the uniquely defined Markov extension of tr_{B_1} to B_2 , that is

$$tr_{B_2}(be_{B_0}) = \beta^{-1}tr_{B_2}(b) \text{ for all } b \in B_1,$$

and let tr_{A_2} be the restriction of this trace to A_2 . Let also tr_{A_1} be the trace on A_1 with Markov extension tr' of modulus γ to $\langle A_1, e_{A_0} \rangle$.

For $a, a' \in A_1$ we get

$$\begin{aligned} tr_{A_2}(ae_{B_0}a') &= tr_{B_2}(a'ae_{B_0}) = \\ \beta^{-1}tr_{B_1}(a'a) &= \beta^{-1}tr_{A_1}(a'a), \end{aligned}$$

and

$$tr'(ae_{A_0}a') = \gamma^{-1}tr_{A_1}(a'a),$$

and hence

$$tr' \circ \phi(ae_{B_0}a') = \gamma^{-1}tr_{A_1}(a'a) = \frac{\beta}{\gamma}tr_{A_2}(ae_{B_0}a').$$

Because ϕ is normal we now have

$$tr' \circ \phi|_{L^2(A_2, tr_{A_2})} = \frac{\beta}{\gamma}tr_{A_2},$$

and the faithfulness of tr' and tr_{A_2} implies that ϕ is injective on zA_1 .

z is the largest projection in $\overline{A_1 e_{B_0} A_1}^{weak}$, and hence $\phi(z) = \mathbf{1}$, the largest projection in $\langle A_1, e_{A_0} \rangle$, and $\phi(\mathbf{1} - z) = 0$.

We have now established

$$\phi|_{zA_2} \text{ is an isomorphism of } zA_2 \text{ onto } \langle A_1, e_{A_0} \rangle \quad (14.2)$$

$$\phi|_{(1-z)A_2} = 0 \quad (14.3)$$

(14.3) implies that $(\mathbf{1} - z)A_2$ is a subalgebra of A_1 since $z = \mathcal{Z}_{A_2}(e_{B_0})$.

Finally we have

$$1 = tr'(1) = tr' \circ \phi(z) = \frac{\beta}{\gamma}tr_{A_2}(z),$$

hence

$$\text{tr}_{A_2}(z) = \frac{\gamma}{\beta},$$

and we must have $\gamma \leq \beta$ with equality if and only if $z = \mathbf{1}$, that is if and only if $A_2 \cong \langle A_1, e_{A_0} \rangle$.

□

Lemma 14.2 *Let $A \subset B \subset \mathcal{B}(L^2(B, \text{tr}_B))$ be finite von Neumann-algebras, and p be a minimal central projection of A . Then $p' = J_B p J_B$ is a minimal central projection in $\langle B, e_A \rangle$ and $e_A p' = e_A p$.*

Proof $p' = J_B p J_B$ is a minimal central projection in $\langle B, e_A \rangle$, since $\langle B, e_A \rangle = J_B A' J_B$. Let ξ be the cyclic and separating trace vector for $L^2(B, \text{tr}_B)$, and let $x \in B$. Then

$$\begin{aligned} e_A J_B p J_B (x\xi) &= e_A x p \xi = & (\text{since } e_A \text{ acts as } E_A \text{ on } B) \\ E_A(xp)\xi &= p E_A(x)\xi = & (\text{since } p \in A' \cap A) \\ p e_A(x\xi) &= e_A p(x\xi) & (\text{since } e_A \in A'). \end{aligned}$$

The density of $B\xi$ in $L^2(B, \text{tr}_B)$ then implies $e_A p' = e_A p$.

□

Lemma 14.3 *If $\mathbf{1} \in A \subset B \subset \mathcal{B}(L^2(B, \text{tr}_B))$ are finite von Neumann-algebras, and p is a minimal central projection of A , and $f \in Ap$ is a projection, then $f e_A$ is a projection in $\langle B, e_A \rangle p'$, where $p' = J_B p J_B$. Furthermore, if f is a minimal projection then $f e_A$ is minimal in $\langle B, e_A \rangle p'$.*

Proof Since $e_A \in A'$, $f e_A$ is a projection.

Let ξ be the cyclic and separating trace vector for $L^2(B, \text{tr}_B)$ and $x \in B$ then

1. $f e_A J_B p J_B (x\xi) = f e_A p x \xi =$ (by lemma 14.2)
 $f E_A(xp)\xi = f E_A(x)p\xi = f p e_A x \xi = f e_A(x\xi)$ (since $f \leq p$)
2. $J_B p J_B f e_A(x\xi) = J_B p J_B f E_A(x)\xi = f E_A(x)p\xi = f p E_A(x)\xi =$
 $f E_A(x)\xi = f e_A(x\xi)$ (since $f \leq p$)

I.e. $f e_A p' = p' f e_A = f e_A \Rightarrow f e_A \leq p'$, hence $f e_A \in \langle B, e_A \rangle p'$.

Assume f is minimal in Ap and that $f_0 \in \langle B, e_A \rangle$ is a projection such that $f_0 \leq f e_A$. Then

$$f_0 \in e_A \langle B, e_A \rangle e_A, \text{ since } f_0 \leq f e_A \in e_A \langle B, e_A \rangle e_A = A e_A,$$

hence we can find a projection $g \in A$, such that $f_0 = g e_A \leq f e_A$, but $A \rightarrow A e_A$ is an isomorphism, so the minimality of f implies $g = f$ or $g = 0$. Consequently $f_0 = 0$ or $f_0 = f e_A$, and $f e_A$ is minimal in $\langle B, e_A \rangle$.

□

Proposition 14.4 Let $\begin{array}{ccc} B_0 & \subset_L & B_1 \\ \cup_K & & \cup_H \\ A_0 & \subset_G & A_1 \end{array}$ be a commuting square of infinite dimensional multi-matrix algebras with respect to tr_{B_1} on B_1 and put $B_2 = \langle B_1, e_{B_0} \rangle$ and $A_2 = \{A_1, e_{B_0}\}''$.

If $\phi : A_2 \rightarrow \langle A_1, e_{A_0} \rangle$ is defined (as in 14.1) by

$$\phi(x) = \pi(x|_{L^2(A_1, tr_{A_1})}), x \in A_2,$$

is an isomorphism with inverse ψ then

1.

$$A_2 \subset_K B_2,$$

Or stated in other terms: For p, q minimal central projections of A_0 resp. B_0 , $q' = \psi(J_{A_1} q J_{A_1})$ and $p' = J_{B_1} p J_{B_1}$ are the corresponding minimal central projections of A_2 resp. B_2 , and

$$[(B_0)_{pq} : (A_0)_{pq}] = [(B_2)_{p'q'} : (A_2)_{p'q'}].$$

2. Assume furthermore that tr_{B_1} is a Markov trace of modulus β for $B_0 \subset B_1$, and the restriction to A_1 , tr_{A_1} , is a Markov trace of modulus γ for $A_0 \subset A_1$ and put

$$B_j = \langle B_{j-1}, e_{B_{j-2}} \rangle \text{ and } A_j = \{A_{j-1}, e_{B_{j-2}}\}'', \text{ for } j \geq 2,$$

then the inclusions are given by :

$B_j \subset B_{j+1}$ is given by L when j is even and L^t when j is odd.

$A_j \subset A_{j+1}$ is given by G when j is even and G^t when j is odd.

$A_j \subset B_j$ is given by K when j is even and H when j is odd.

Proof Let f be a minimal projection in $A_0 q$, and let $pf = \sum_{i=1}^n g_i$ be a decomposition into minimal orthogonal projections in $B_0 p$. I.e.

$$K_{pq} = n = [(B_0)_{pq} : (A_0)_{pq}].$$

By lemma 14.3 $f e_{A_0}$ is minimal in $\langle A_1, e_{A_0} \rangle_{J_{A_1} q J_{A_1}}$, hence $\psi(f e_{A_0}) = f e_{B_0}$ is minimal in $A_2 q'$. By lemma 14.1 we then have

$$\begin{aligned} f e_{B_0} p' &= \\ f e_{B_0} p &= \\ f p e_{B_0} &= \quad (\text{since } p \in B_0 \text{ and } e_{B_0} \in B_0') \\ p f e_{B_0} &= \\ \sum_{i=1}^n g_i e_{B_0}. & \end{aligned}$$

I.e. $fe_{B_0}p'$ is a sum of n minimal orthogonal projections in $(B_2)_{p'q'}$, and hence $A_2 \subset_K B_2$.

By assumption $A_2 \cong \langle A_1, e_{A_0} \rangle$, so lemma 14.1 yields

$$\mathcal{Z}_{A_2}(e_{B_0}) = \mathbf{1} \Leftrightarrow \beta = \gamma.$$

Also by assumption and (1) we have the commuting squares

$$\begin{array}{ccccc} B_0 & \subset_L & B_1 & \subset_{L^t} & B_2 \\ \cup_K & & \cup_H & & \cup_K \\ A_0 & \subset_G & A_1 & \subset_{G^t} & A_2 \end{array}$$

and hence the extension tr_{B_2} of tr_{B_1} is a Markov trace for $B_1 \subset B_2$, and $A_2 \cong \langle A_1, e_{A_0} \rangle$ also implies that the restriction of tr_{B_2} to A_2 is a Markov trace since:

For $a \in A_1$ we have

$$\begin{aligned} tr_{\langle A_1, e_{A_0} \rangle}(\psi(ae_{A_0})) &= tr_{A_2}(ae_{B_0}) = \\ tr_{B_2}(ae_{B_0}) &= \beta^{-1}tr_{A_1}(a) = \beta^{-1}tr_{B_1}(a). \end{aligned}$$

Assume now that $\phi : A_{j+1} \rightarrow \langle A_j, e_{A_{j-1}} \rangle$ is an isomorphism for some

$j \geq 2$, where $A_j = \{A_{j-1}, e_{B_{j-2}}\}''$ is defined inductively. We then have the following picture

$$\begin{array}{ccccc} B_j & \subset & B_{j+1} & \subset & B_{j+2} \\ \cup & & \cup & & \cup \\ A_j & \subset & A_{j+1} & \subset & A_{j+2} \end{array},$$

with $A_{j+1} \cong \langle A_j, e_{A_{j-1}} \rangle$.

According to lemma 14.1

$$A_{j+2} = zA_{j+2} \oplus (\mathbf{1} - z)A_{j+2}, \text{ with } z = \mathcal{Z}_{A_{j+2}}(e_{B_j}).$$

Since z is central, zA_{j+2} is a two-sided ideal in A_{j+2} , and lemma 14.1 also yields that $A_{j+1}e_{B_j}A_{j+1}$ is a dense $*$ -subalgebra. In particular

$$\beta e_{B_{j-1}}e_{B_j}e_{B_{j-1}} = e_{B_j} \in zA_{j+2},$$

that is

$$zA_{j+2} \supset A_j e_{B_{j-1}} A_j,$$

and since zA_{j+2} is weakly closed, we have

$$zA_{j+2} \cong \overline{A_j e_{B_{j-1}} A_j}^{weak} = \{A_j, e_{B_{j-1}}\}'' \ni \mathbf{1}.$$

Hence $z = \mathbf{1}$ and $A_{j+2} = zA_{j+2} \cong \langle A_{j+1}, e_{A_j} \rangle$, and the statements concerning the inclusion patterns follows from the first part and induction.

□

Corollary 14.5 *Let $\Gamma_1, \Gamma_2, \Gamma_3$, and Γ_4 be finite or infinite, locally finite bi-partite graphs with adjacency matrices of a bi-partition G, H, K resp. L and Perron-Frobenius vectors ξ_1, ξ_2, ξ_3 resp. ξ_4 . If*

$$\begin{array}{ccc} B_0 & \subset_L & B_1 \\ \cup_K & & \cup_H \\ A_0 & \subset_G & A_1 \end{array} \quad (14.4)$$

is a symmetric commuting square with respect to the finite trace, tr_{B_1} , on B_1 given by the corresponding partition of ξ_2 resp. ξ_4 , we define inductively

$$B_j = \langle B_{j-1}, e_{B_{j-2}} \rangle \text{ and } A_j = \{A_{j-1}, e_{B_{j-2}}\}'', \text{ for } j \geq 2.$$

Then

$$\begin{array}{ccc} B_j & \subset & B_{j+1} \\ \cup & & \cup \\ A_j & \subset & A_{j+1} \end{array}$$

is a symmetric commuting square for each j . And we obtain the ladder

$$\begin{array}{ccccccc} B_0 & \subset_L & B_1 & \subset_{L^t} & B_2 & \subset_L & B_3 & \cdots \\ \cup_K & & \cup_H & & \cup_K & & \cup_H & \\ A_0 & \subset_G & A_1 & \subset_{G^t} & A_2 & \subset_G & A_3 & \cdots \end{array}$$

of multi-matrix algebras.

Proof Since the square (14.4) is symmetric, tr_{B_1} is a Markov trace for $B_0 \subset B_1$ of modulus $\|LL^t\|$ and the restriction to A_1 is a Markov trace of modulus $\|GG^t\|$ for $A_0 \subset A_1$, and we have $\|LL^t\| = \|GG^t\|$. In the terminology of lemma 14.1 $\beta = \gamma$ and hence $A_2 \cong \langle A_1, e_{A_0} \rangle$. We then get the ladder of multi-matrix algebras by proposition 14.4.

□

15 The Limit of the Algebras

15.1 Extremality of the Trace

Let Γ be an infinite, locally finite bi-partite graph with Perron-Frobenius vector ξ , with corresponding eigenvalue λ . Let G be the adjacency matrix of a bi-partition of Γ and let ξ_1, ξ_2 be the corresponding splitting of ξ .

Assume that

$$A_0 \subset_G A_1 \subset_{G^t} A_2 \subset_G A_3 \subset_{G^t} A_4 \dots$$

is a tower of multi-matrix algebras, and that the obvious trace, tr_n , on the A_n 's defined by the vectors $\lambda^{-(n-1)}\xi_1$ when n is odd, and $\lambda^{-(n-1)}\xi_2$ when n is even, is finite.

The induced trace on $A_\infty = \cup_{n=1}^\infty A_n$ is denoted by tr .

Assume now that ω_n is another trace on the A_n 's (extending one another) with induced trace ω on A_∞ , with the property that $0 < \omega_n \leq tr_n$ for all n . Let ω_n be given by the vector η_n . The assertion $0 < \omega_n \leq tr_n$ is equivalent to $0 < \eta_n \leq \lambda^{-(n-1)}\xi_i$ where $i = 1$ if n is odd and $i = 2$ if n is even. I.e. we must have

$$\eta_{2n} \leq \lambda^{-(2n-1)}\xi_2$$

and

$$\eta_{2n+1} \leq \lambda^{-2n}\xi_1$$

for all n . In particular

$$\|\eta_{2n}\|_2 \leq \lambda^{-(2n-1)}\|\xi_2\|_2 \leq \lambda^{-(2n-1)}\|\xi\|_2$$

and

$$\|\eta_{2n+1}\|_2 \leq \lambda^{-2n}\|\xi_1\|_2 \leq \lambda^{-2n}\|\xi\|_2.$$

I.e.

$$\|\eta_k\|_2 \leq \lambda^{-k+1}\|\xi\|_2.$$

The extension property of the ω_n 's is stated as

$$\eta_1 = GG^t\eta_3 = (GG^t)^2\eta_5 = (GG^t)^3\eta_7 \dots$$

and

$$\eta_2 = G^tG\eta_4 = (G^tG)^2\eta_6 = (G^tG)^3\eta_8 \dots$$

Consider $k = 2n + 1, n \in \mathbb{N}$. Let ϕ_k equal the projection of η_k on ξ_1 , and put

$$\psi_k = \eta_k - \phi_k.$$

Since GG^t leaves $\mathbb{C}\xi_1$ and its orthogonal complement invariant, we have

$$GG^t\phi_k = \phi_{k-2} \text{ and } GG^t\psi_k = \psi_{k-2}.$$

Consider the functions $f_n(t) = t^n, t \in [0, 1]$.

Since $\|f_n\|_\infty \leq 1$ and $f_n \xrightarrow{n \rightarrow \infty} \chi_{\{1\}}$ pointwise, and we have

$$f_n\left(\frac{1}{\lambda^2}GG^t\right) \xrightarrow{n \rightarrow \infty} \chi_{\{1\}}\left(\frac{1}{\lambda^2}GG^t\right) \text{ strongly.}$$

I.e.

$$f_n\left(\frac{1}{\lambda^2}GG^t\right) \xrightarrow{n \rightarrow \infty} \text{the projection onto } \xi_1 \quad (15.1)$$

We now have

$$\|\psi_k\|_2 = \|(GG^t)^n\psi_{k+2n}\|_2 \rightarrow 0 \text{ as } n \rightarrow \infty$$

hence $\|\psi_k\|_2 = 0$ for all odd k

Similarly we get $\|\psi_k\|_2 = 0$ for all even k .

I.e. $\eta_k = c_k\xi_1$ for k odd and $\eta_k = c_k\xi_2$ for k even.

Let k be odd, then

$$GG^t\eta_k = \eta_{k-2} = c_{k-2}\xi_1$$

and

$$GG^t\eta_k = \lambda^2 c_k \xi_1$$

hence $c_{k-2} = \lambda^2 c_k$ which yields $c_{2n+1} = \lambda^{-2n} c_1$

The same way we get $c_{2n} = \lambda^{-2n+1} c_2$

From $G\eta_2 = \eta_1 = c_1 \xi_1$ and $Gc_2 \xi_2 = \lambda c_2 \xi_1$ we also have $c_2 = \lambda^{-1} c_1$

Hence $tr' = c_1 tr$, implying that tr is extremal.

15.2 Construction of Subfactors

Let $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ be bi-partite, locally finite graphs, and G, H, K and L the adjacency matrices of a bi-partition. Let $\begin{array}{ccc} B_0 & \subset_L & B_1 \\ \cup_K & & \cup_H \\ A_0 & \subset_G & A_1 \end{array}$ be a symmetric commuting square with respect to the finite trace defined by the Perron- Frobenius vector of $L^t L$ and its restrictions to the other multi-matrix algebras. Since the square is a symmetric commuting square $\|GG^t\| = \|LL^t\|$, so by corollary 14.5, and the construction herein, we get the infinite ladder of multi-matrix algebras

$$\begin{array}{ccccccc} B_0 & \subset_L & B_1 & \subset_{L^t} & B_2 & \subset_L & B_3 & \cdots \\ \cup_K & & \cup_H & & \cup_K & & \cup_H & \\ A_0 & \subset_G & A_1 & \subset_{G^t} & A_2 & \subset_G & A_3 & \cdots \end{array}$$

with traces tr_{A_n} and tr_{B_n} extending each other.

The induced trace on the inductive limit $B_\infty = \bigcup_{n=1}^\infty B_n$ is denoted by tr_{B_∞} . Put $A_\infty = \bigcup_{n=1}^\infty A_n$, the inductive limit of the A_n 's, with limit of traces denoted by tr_{A_∞} . Then $A_\infty \subset B_\infty$, and tr_{B_∞} extends tr_{A_∞} . Let B equal the weak closure of the G-N-S-representation of B_∞ , and A equal the weak closure of A_∞ in B . Then $A \subset B$.

By Kaplansky's density theorem $Unitball(B) = \overline{Unitball(B_\infty)}^{2-norm}$. The unitball of the weak closure of A_∞ in B is, according to Kaplansky, equal to $\overline{Unitball(A_\infty)}^{2-norm}$, and since tr_{B_∞} extends tr_{A_∞} the weak closure of A_∞ in B equals the weak closure of A_∞ in the G-N-S-representation of A_∞ with respect to tr_{A_∞} . The previous part of this section showed that the traces tr_{A_∞} resp. tr_{B_∞} are extremal, and hence $A \subset B$ are hyperfinite II_1 -factors with traces tr_A resp. tr_B .

From now on consider the A_n 's and the B_n 's as algebras represented on $L^2(B_\infty, tr_{B_\infty}) = L^2(B, tr_B)$, we then have the following

Lemma 15.1 *Let $e : L^2(B, tr_B) \rightarrow L^2(A, tr_A)$ be the orthogonal projection. For all n the restriction of e to $L^2(B_n, tr_{B_n})$, $e|_{L^2(B_n, tr_{B_n})}$, equals $e_n : L^2(B_n, tr_{B_n}) \rightarrow L^2(A_n, tr_{A_n})$.*

Proof The commuting square condition implies that e_{n+1} extends e_n , hence we can define a

surjection

$$f : \bigcup_{n=1}^{\infty} L^2(B_n, tr_{B_n}) \rightarrow \bigcup_{n=1}^{\infty} L^2(A_n, tr_{A_n})$$

by

$$f|_{L^2(B_n, tr_{B_n})} = e_n.$$

Then f is linear, $f^2 = f$, $\|f\| \leq 1$ and $(fx, y) = (x, fy)$ for all

$x, y \in \bigcup_{n=1}^{\infty} L^2(B_n, tr_{B_n})$. Hence f has a unique extension to

$$g : \overline{\bigcup_{n=1}^{\infty} L^2(B_n, tr_{B_n})} \rightarrow \overline{\bigcup_{n=1}^{\infty} L^2(A_n, tr_{A_n})}$$

$g^2 = g$ and $g^* = g$, hence g is an orthogonal projection, and its image is closed and contains $\bigcup_{n=1}^{\infty} L^2(A_n, tr_{A_n})$. This now implies that g is onto $\overline{\bigcup_{n=1}^{\infty} L^2(A_n, tr_{A_n})}$.

Since $\bigcup_{n=1}^{\infty} B_n$ is dense in B in the $\|\cdot\|_2$ -norm we have

$$L^2(B, tr_B) \supset \overline{\bigcup_{n=1}^{\infty} L^2(B_n, tr_{B_n})} \supset L^2(B, tr_B)$$

hence

$$L^2(B, tr_B) = \overline{\bigcup_{n=1}^{\infty} L^2(B_n, tr_{B_n})}.$$

Similarly we get

$$L^2(A, tr_A) = \overline{\bigcup_{n=1}^{\infty} L^2(A_n, tr_{A_n})}.$$

I.e. g is the orthogonal projection of $L^2(B, tr_B)$ onto $L^2(A, tr_A)$, that is $g = e$, and hence $e|_{L^2(B_n, tr_{B_n})} = e_n$ for all n .

□

16 A Trace on $\langle B, e \rangle$ and the Index

Assume the symmetric commuting squares of infinite dimensional multi-matrix algebras

$$\begin{array}{ccccccc} B_0 & \subset_L & B_1 & \subset_{L^t} & B_2 & \subset_L & B_3 & \cdots \\ \cup_K & & \cup_H & & \cup_K & & \cup_H & \\ A_0 & \subset_G & A_1 & \subset_{G^t} & A_2 & \subset_G & A_3 & \cdots \end{array}$$

all algebras considered represented on $L^2(B, tr_B)$, and let e denote the orthogonal projection $e : L^2(B, tr_B) \rightarrow L^2(A, tr_A)$, A and B as previously.

By lemma 15.1 $e|_{L^2(B_n, tr_{B_n})} = e_n$ is the fundamental projection of

$A_n|_{L^2(B_n, tr_{B_n})} \subset B_n|_{L^2(B_n, tr_{B_n})}$ hence $\langle B_n, e \rangle \cong \langle B_n|_{L^2(B_n, tr_{B_n})}, e_n \rangle$. Let p_n denote the orthogonal projection of $L^2(B, tr_B)$ onto $L^2(B_n, tr_{B_n})$, then p_n is the projection corresponding to the fundamental construction for $B_n \subset B$, and $ep_n = p_ne$ since $\begin{array}{ccc} B_n & \subset & B \\ \cup & & \cup \\ A_n & \subset & A \end{array}$ is a commuting square.

$\langle B_n, e \rangle$ has a unique normal trace $tr_{\langle B_n, e \rangle}$ such that

$$tr_{\langle B_n, e \rangle}(xe) = \beta^{-1}tr_{B_n}(x) \text{ for all } x \in B_n,$$

where β is the Perron-Frobenius eigenvalue of HH^t . We will now show that for $x \in \langle B, e \rangle$,

$$p_nxp_n|_{L^2(B_n, tr_{B_n})} \in \langle B_n, e \rangle|_{L^2(B_n, tr_{B_n})}.$$

To prove this it is enough to consider $x \in B \cup eBe$, which is a dense subalgebra of $\langle B, e \rangle$.

1. $x = b \in B$.

$$\begin{aligned} p_nb p_n|_{L^2(B_n, tr_{B_n})} &= E_{B_n}(b)p_n|_{L^2(B_n, tr_{B_n})} = \\ E_{B_n}(b)|_{L^2(B_n, tr_{B_n})} &\in B_n|_{L^2(B_n, tr_{B_n})}, \end{aligned}$$

where E_{B_n} is the trace preserving conditional expectation of B onto B_n .

2. $x = ebe, b \in B$.

$$\begin{aligned} p_n e b e p_n|_{L^2(B_n, tr_{B_n})} &= \\ e p_n b p_n e|_{L^2(B_n, tr_{B_n})} &= \\ e E_{B_n}(b) e p_n|_{L^2(B_n, tr_{B_n})} &= \\ e E_{B_n}(b)|_{L^2(B_n, tr_{B_n})} &\in \langle B_n, e \rangle|_{L^2(B_n, tr_{B_n})}. \end{aligned}$$

For $n \in \mathbb{N}$ define a positive normal state τ_n on $\langle B, e \rangle$ by

$$\tau_n(x) = tr_{\langle B_n, e \rangle}(p_nxp_n|_{L^2(B_n, tr_{B_n})}) \text{ for } x \in \langle B, e \rangle.$$

We will show that τ_n is independent of n .

Again it is enough to consider $x \in B \cup eBe$

1. $x = b \in B$.

$$\tau_n(b) = tr_{B_n}(E_{B_n}(b)) = tr_B(b)$$

2. $x = ebe, b \in B$.

$$\begin{aligned} \tau_n(x) &= tr_{\langle B_n, e \rangle}(e E_{B_n}(b) e) = \\ \beta^{-1}tr_{B_n}(E_{B_n}(b)) &= \beta^{-1}tr_B(b) \end{aligned}$$

I.e. τ_n is independent of n .

On $\bigcup_{n=1}^{\infty} \langle B_n, e \rangle$, which is weakly dense in $\langle B, e \rangle$, put $\tau = \tau_n$ “for all n ”.

For $b \in B_n$ we get,

1.

$$\tau(b) = \tau_n(b) = \text{tr}_{B_n}(E_{B_n}(b)) = \text{tr}_{B_n}(b) = \text{tr}_{\langle B_n, e \rangle}(b)$$

and

2.

$$\tau(ebe) = \tau_n(ebe) = \text{tr}_{\langle B_n, e \rangle}(eE_{B_n}(b)e) = \text{tr}_{\langle B_n, e \rangle}(ebe),$$

i.e. τ extends all the $\text{tr}_{\langle B_n, e \rangle}$'s, and hence τ is a trace on $\langle B, e \rangle$ extending the trace tr_B on B .

For any $b \in B_n$ we have

$$\tau(be) = \text{tr}_{\langle B_n, e \rangle}(eE_{B_n}(b)e) = \beta^{-1} \text{tr}_{B_n}(E_{B_n}(b)) = \beta^{-1} \text{tr}_B(b),$$

and since $B = \overline{\bigcup_{n=1}^{\infty} B_n}^{\text{weak}}$ we get $\tau(be) = \beta^{-1} \text{tr}_B(b)$ for all $b \in B$. I.e. τ is a Markov extension of tr_B to $\langle B, e \rangle$ of modulus β .

The index $[B : A] = [\langle B, e \rangle : B]$ is now determined as $\tau(e)^{-1} = \beta$.

17 The Dimension of the Relative Commutant of A in B

Let Γ be an infinite, locally finite bi-partite graph with Perron-Frobenius vector ξ , G the adjacency matrix of a bi-partition, and ξ_1, ξ_2 the corresponding splitting of ξ .

Assume that

$$A_0 \subset_G A_1 \subset_{G^t} A_2 \subset_G A_3 \cdots$$

is a tower of multi-matrix algebras, with finite trace defined by ξ that is, if say ξ_1 defines the trace on A_0

$$\text{tr}(\mathbf{1}) = \sum_{j=1}^{\infty} a_j^0 (\xi_1)_j < \infty$$

where $a_n = (a_i^n)_{i=1}^{\infty}$ is the dimension vector of A_n . We then have the following proposition

Proposition 17.1 *In the above situation put ξ_1 equal to the Perron-Frobenius vector of GG^t and ξ_2 equal to the Perron-Frobenius vector of G^tG . Then the dimension vectors converge pointwise to a multiple of ξ_1 when n is even resp. a multiple of ξ_2 when n is odd.*

Proof Since $a_j^0 > 1$ we must have $\|\xi_1\|_1 < \infty$.

Formally the dimension of the i 'th summand in A_{2n} is given as

$$\langle (GG^t)^n a_0, \delta_i \rangle, a_0 = \begin{pmatrix} a_1^0 \\ a_2^0 \\ \vdots \end{pmatrix}, (\delta_i)_j = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}$$

Fubini's theorem for positive functions gives

$$\langle (\frac{1}{\lambda^2} GG^t)^n a_0, \delta_i \rangle = \langle a_0, (\frac{1}{\lambda^2} GG^t)^n \delta_i \rangle$$

Choose ξ_1 with $\|\xi_1\|_2 = 1$, and let ξ_1, x_1, x_2, \dots be an orthonormal basis for $l^2(\mathbb{N})$ then

$$(\frac{1}{\lambda^2} GG^t)^n \delta_i = \langle (\frac{1}{\lambda^2} GG^t)^n \delta_i, \xi_1 \rangle \xi_1 + \sum_{j=1}^{\infty} \langle (\frac{1}{\lambda^2} GG^t)^n \delta_i, x_j \rangle x_j$$

and

$$\begin{aligned} & \| (\frac{1}{\lambda^2} GG^t)^n \delta_i - (\xi_1)_i \xi_1 \|_2^2 = \\ & \| \langle (\frac{1}{\lambda^2} GG^t)^n \delta_i, \xi_1 \rangle \xi_1 - (\xi_1)_i \xi_1 \|_2^2 + \sum_{j=1}^{\infty} \| \langle (\frac{1}{\lambda^2} GG^t)^n \delta_i, x_j \rangle x_j \|_2^2 \end{aligned}$$

Each $x_j \in (\mathbb{C}\xi_1)^\perp$ so by (15.1) we get $(\frac{1}{\lambda^2} GG^t)^n x_j \xrightarrow{n \rightarrow \infty} 0$ for any j . Hence

$$\| \langle (\frac{1}{\lambda^2} GG^t)^n \delta_i, x_j \rangle x_j \|_2 = \| \langle \delta_i, (\frac{1}{\lambda^2} GG^t)^n x_j \rangle x_j \|_2 \xrightarrow{n \rightarrow \infty} 0$$

and

$$\lim_{n \rightarrow \infty} \| (\frac{1}{\lambda^2} GG^t)^n \delta_i - (\xi_1)_i \xi_1 \|_2 = \lim_{n \rightarrow \infty} \| \langle (\frac{1}{\lambda^2} GG^t)^n \delta_i, \xi_1 \rangle \xi_1 - (\xi_1)_i \xi_1 \|_2$$

For any n we have

$$\langle (\frac{1}{\lambda^2} GG^t)^n \delta_i, \xi_1 \rangle = \langle \delta_i, (\frac{1}{\lambda^2} GG^t)^n \xi_1 \rangle = \langle \delta_i, \xi_1 \rangle = (\xi_1)_i$$

i.e.

$$(\frac{1}{\lambda^2} GG^t)^n \delta_i \xrightarrow{\|\cdot\|_2} (\xi_1)_i \xi_1 \text{ as } n \rightarrow \infty.$$

We have $\delta_i \leq \frac{1}{(\xi_1)_i} \xi_1$, and since $\|\xi_1\|_1 < \infty$, $\frac{1}{(\xi_1)_i} \xi_1$ is summable, and we get

$$\langle a_0, (\frac{1}{\lambda^2} GG^t)^n \delta_i \rangle = \sum_{j=1}^{\infty} a_j^0 \langle (\frac{1}{\lambda^2} GG^t)^n \delta_i, x_j \rangle \xrightarrow{n \rightarrow \infty} \sum_{j=1}^{\infty} a_j^0 (\xi_1)_j (\xi_1)_i = \text{tr}(\mathbf{1})(\xi_1)_i$$

A similar argument holds for the odd labeled floors of the tower

□

Remark 17.2 If we are in the above situation then the trace vector α_n of A_n is given as

$$\alpha_{2k} = \frac{1}{\lambda^{2k}} \xi_1, \quad \alpha_{2k+1} = \frac{1}{\lambda^{2k+1}} \xi_2$$

and we have

$$\|\xi_2\|_2 = \langle \xi_2, \xi_2 \rangle = \frac{1}{\lambda^2} \langle G^t \xi_1, G^t \xi_1 \rangle = \|\xi_1\|_2$$

Let z_i^k be the minimal central projection in A_k corresponding to the i 'th component. Then

$$\begin{aligned} \text{tr}(z_i^{2l}) &= (\alpha_{2l})_i (a_{2l})_i \\ &= \lambda^{2l} (\alpha_{2l})_i \frac{1}{\lambda^{2l}} (a_{2l})_i \\ &= (\xi_1)_i \left(\left(\frac{1}{\lambda^2} G G^t \right)^l \mathbf{1} \right)_i \\ &\xrightarrow{n \rightarrow \infty} (\xi_1)_i (\xi_1)_i \text{tr}(\mathbf{1}) > 0. \end{aligned}$$

In particular there exists a constant c_i independent of l s.t.

$$\text{tr}(z_i^{2l}) \geq c_i \text{ for all } l$$

A similar argument holds for the odd labeled A_n 's.

The following is essentially contained in [Wen1]

Lemma 17.3 *Let $\{\alpha_1, \dots, \alpha_m\}$ be m different real numbers, and $\{t_1, \dots, t_m\}$ be positive numbers with sum 1. Then there exists $\epsilon > 0$ such that:*

For any II_1 -factor A and two selfadjoint elements $a, b \in A$ satisfying

1. $a = \sum_{i=1}^m \alpha_i p_i$, $\{p_i\}$ orthogonal projections with $\text{tr}(p_i) = t_i$
2. b has strictly less than m spectral values, and $\|b\| \leq \|a\|$

then $\|a - b\|_2^2 \geq \epsilon$

Proof Let $k < m$ and put

$$\mathcal{K} = \left\{ (\beta, V) \left| \begin{array}{l} \beta = (\beta_1, \dots, \beta_k), \\ 0 \leq \beta_j \leq \max\{|\alpha_i|\}, \\ V = (v_{ij}) \in M_{m \times k}([0, 1]), \sum_j v_{ij} = t_i \end{array} \right. \right\}$$

Then \mathcal{K} is compact and

$$F(\beta, V) = \sum_{i,j} (\alpha_i - \beta_j)^2 v_{ij}$$

has a minimum on \mathcal{K} . Assume this minimum is attained at (β', V') . Since $k < m$ there exists i s.t. $\alpha_i \notin (\beta'_1, \dots, \beta'_k)$ and $t_i > 0$ implies that at least one $v'_{ij} > 0$. Hence $F(\beta', V') > 0$.

Put $\epsilon = F(\beta', V')$, and let A be a II_1 -factor, and let $a, b \in A_{sa}$ satisfy (1) and (2).

Let $b = \sum_{j=1}^k \beta_j q_j$ be the spectral decomposition of b . Since $\sum p_j = \mathbf{1} = \sum q_i$ we have

$$a = \sum_{i,j} \alpha_i p_i q_j \text{ and } b = \sum_{i,j} \beta_j p_i q_j$$

i.e.

$$(a - b) = \sum_{i,j} (\alpha_i - \beta_j) p_i q_j \text{ and}$$

$$(a - b)^2 =$$

$$(a - b)(a - b)^* =$$

$$\sum_{i,j} (\alpha_i - \beta_j)^2 p_i q_j p_i +$$

$$\sum_{i,j,i',j', i \neq i', j \neq j'} (\alpha_i - \beta_j)(\alpha_{i'} - \beta_{j'}) p_i q_j q_{j'} p_{i'} \quad .$$

Hence

$$tr((a - b)^2) = \sum_{i,j} (\alpha_i - \beta_j)^2 tr(p_i q_j p_i)$$

since $tr(p_i q_j p_{i'} q_{j'}) = 0$ when $(i, j) \neq (i', j')$.

Since

$$0 \leq tr(p_i q_j p_i) \leq \sum_j tr(p_i q_j p_i) = tr(p_i) = t_i$$

$W = (w_{ij}), w_{ij} = tr(p_i q_j p_i)$ is a matrix of the type defining the second coordinate of \mathcal{K} , and we have obtained

$$\|a - b\|_2^2 = tr((a - b)^2) = F(\beta, W) \geq F(\beta', V') = \epsilon$$

□

Theorem 17.4 *If $\Gamma_G, \Gamma_H, \Gamma_K, \Gamma_L$ are finite or infinite, locally finite bi-partite graphs with Perron-Frobenius vectors $\xi_1, \xi_2, \xi_3, \xi_4$ defining finite traces on the ladder*

$$\begin{array}{ccccccc} B_0 & \subset_L & B_1 & \subset_{L^t} & B_2 & \subset_L & B_3 & \cdots \\ \cup_K & & \cup_H & & \cup_K & & \cup_H & \\ A_0 & \subset_G & A_1 & \subset_{G^t} & A_2 & \subset_G & A_3 & \cdots \end{array}$$

of multi-matrix algebras, and $A \subset B$ are II_1 -factors constructed from the tower in the usual way. Then

$$\dim\{A' \cap B\} \leq (\min\{1\text{-norm of rows of } K \text{ and } H\})^2$$

Proof Denote the steps of the ladder by $\bigoplus_i A_i^n$ and $\bigoplus_j B_j^n$ with dimension vectors $a_n = (a_i^n)$ and $b_n = (b_j^n)$.

Assume n is even $n = 2l$. Put $m_0 = \min\{ \text{1-norm of rows of } K \}$ and choose i_0 s.t. the i_0 'th row of k has $\| \cdot \|_1 = m_0$

Let $z_{i_0}^{2l}$ be the corresponding minimal central projection in A_{2l} .

$$z_{i_0}^{2l} \in \mathcal{Z}(A_{2l}) \subset A'_{2l} \cap B_{2l}$$

Let q_j^{2l} be the minimal central projection in B_{2l} corresponding to B_j^{2l}

$A_{2l}q_j^{2l}$ is of the form

$$\underbrace{\begin{pmatrix} x_1 & & & \\ & \ddots & & \\ & & x_1 & \\ & & & x_2 \\ & & & & \ddots \end{pmatrix}}_{b_j^{2l}}$$

where $x_i \in A_i^{2l}$ is repeated K_{ij} -times. In this setting $z_{i_0}^{2l}q_j^{2l}$ has

$$x_i = \begin{cases} \mathbf{1}_{a_{i_0}^{2l} \times a_{i_0}^{2l}} & \text{if } i = i_0 \\ 0 & \text{otherwise.} \end{cases}$$

We also have

$$A'_{2l} \cap B_{2l} \cong \bigoplus_{i,j} M_{K_{ij}}(\mathbb{C})$$

and $z_{i_0}^{2l}$ corresponds to the identity in

$$\bigoplus_j M_{K_{i_0 j}}(\mathbb{C})$$

and hence $z_{i_0}^{2l}$ can be split in m_0 orthogonal projections in $A'_{2l} \cap B_{2l}$

Let $\{p_j \mid j = 1, \dots, m\}$ be a maximal splitting of the identity in $A' \cap B$ into minimal non-trivial projections, and put

$$x = \sum_{j=1}^m \frac{j}{m} p_j.$$

Then x is selfadjoint and $\|x\| = 1$. By Kaplansky's density theorem there exists

$$(x_n) \subset (\cup B_n)_{sa}, \quad \|x_n\| \leq 1, \quad x_n \xrightarrow{n \rightarrow \infty} x \text{ strongly}$$

i.e. $\text{dist}_2((\cup B_n)_1, x) = 0$. Since $B_0 \subset B_1 \subset \dots$ x_n can be chosen in B_n .

In particular $x_{2l} \xrightarrow{l \rightarrow \infty} x$ strongly. Put $y_{2l} = E_{A'_{2l} \cap B_{2l}}(x_{2l}) = E_{A'_{2l} \cap B}(x_{2l})$ then

$$\|y_{2l} - x\|_2 \leq \|y_{2l} - E_{A'_{2l} \cap B}(x)\|_2 + \|E_{A'_{2l} \cap B}(x) - x\|_2.$$

Since $A'_{2l} \cap B \xrightarrow{l \rightarrow \infty} A' \cap B$ we get

$$\|E_{A'_{2l} \cap B}(x) - x\|_2 \xrightarrow{l \rightarrow \infty} \|E_{A' \cap B}(x) - x\|_2 = 0$$

and

$$\|y_{2l} - E_{A'_{2l} \cap B}(x)\|_2 = \|E_{A'_{2l} \cap B}(x_{2l} - x)\|_2 \leq \|x_{2l} - x\|_2 \xrightarrow{l \rightarrow \infty} 0.$$

This shows

$$y_{2l} \xrightarrow{l \rightarrow \infty} x \text{ strongly}$$

$y_{2l} z_{i_0}^{2l}$ has at most m_0 spectral projections, since:

$y_{2l} \in A'_{2l} \cap B_{2l}$ and $z_{i_0}^{2l} \in A_{2l}$ hence $[y_{2l}, z_{i_0}^{2l}] = 0$. In particular

$$y_{2l} z_{i_0}^{2l} \in z_{i_0}^{2l} B_{2l} z_{i_0}^{2l} \cong \bigoplus_j M_{K_{i_0 j}}(\mathbb{C})$$

which only contains m_0 minimal projections.

Since $z_{i_0}^{2l} \in A$ and $p_j \in A' \cap B$ we have $tr_B(p_j z_{i_0}^{2l}) = tr_B(p_j) tr_B(z_{i_0}^{2l}) \neq 0$. I.e. $p_j z_{i_0}^{2l} = z_{i_0}^{2l} p_j$ is a non-zero projection, and we get

$$x z_{i_0}^{2l} = \sum_{j=1}^m \frac{j}{m} p_j z_{i_0}^{2l}$$

has exactly m_0 spectral projections.

Assume $m > m_0$.

In lemma 17.3 put $\alpha_j = \frac{j}{m}, t_j = tr_B(p_j)$ and let the II_1 -factor be $z_{i_0}^{2l} B z_{i_0}^{2l}$. Then there exists $\epsilon > 0$ s.t. for all $y \in z_{i_0}^{2l} B z_{i_0}^{2l}$ selfadjoint with

$\|y\| \leq \|x z_{i_0}^{2l}\|$ and y less than m spectral projections

$$\|y - x z_{i_0}^{2l}\|_{2, z_{i_0}^{2l} B z_{i_0}^{2l}}^2 \geq \epsilon.$$

The trace on $z_{i_0}^{2l} B z_{i_0}^{2l}$ is given by

$$tr_{z_{i_0}^{2l} B z_{i_0}^{2l}}(\cdot) = \frac{tr_B(\cdot)}{tr_B(z_{i_0}^{2l})}$$

hence

$$\|y - x z_{i_0}^{2l}\|_{2, B}^2 \geq \epsilon tr_B(z_{i_0}^{2l}).$$

$\|x z_{i_0}^{2l}\| = 1$ and $\|y_{2l} z_{i_0}^{2l}\| \leq \|x_{2l}\| \leq 1$ and we get

$$\|y_{2l} z_{i_0}^{2l} - x z_{i_0}^{2l}\|_{2, B}^2 \geq \epsilon tr_B(z_{i_0}^{2l}) \geq \epsilon c_{i_0} > 0$$

where c_{i_0} is the constant discussed in remark 17.2.

On the other hand we have

$$\|y_{2l}z_{i_0}^{2l} - xz_{i_0}^{2l}\|_{2,B} \leq \|z_{i_0}^{2l}\|_{2,B} \|y_{2l} - x\|_{2,B} \xrightarrow{l \rightarrow \infty} 0.$$

This is a contradiction. I.e. $m \leq m_0$. Since

$$\sum_j K_{i_0,j}^2 \leq \left(\sum_j K_{i_0,j} \right)^2$$

we have

$$\dim\{A' \cap B\} \leq (\min\{ \text{1-norm of rows of } K \})^2$$

The same argument holds for odd n 's involving the matrix H instead. And we get

$$\dim\{A' \cap B\} \leq (\min\{ \text{1-norm of rows of } K \text{ and } H \})^2,$$

because

$$\dim\{A' \cap B\} \leq (\dim\{ \text{maximal Abelian subalgebra of } A' \cap B \})^2 = m^2.$$

□

Corollary 17.5 *If there exists a symmetric commuting square*

$$\begin{array}{ccc} B_0 & \subset_L & B_1 \\ \cup_K & & \cup_H \\ A_0 & \subset_G & A_1 \end{array}$$
of infinite dimensional multi-matrix algebras, then there exists a pair of hyperfinite II_1 -factors $A \subset B$ with

$$\dim\{A' \cap B\} \leq (\min\{ \text{1-norm of rows and columns of } K \text{ and } H \})^2$$

Proof If the minimum is attained for a row of either H or K , the result follows from Theorem 17.4. If the minimum is attained for a column of either H or K , we argue as follows.

Let e_{A_1} denote the projection in the basic construction for $A_1 \subset B_1$, $C_1 = \langle B_1, e_{A_1} \rangle$ and $C_0 = \langle B_0, e_{A_1} \rangle$. Then

$$\begin{array}{ccc} C_0 & \subset_G & C_1 \\ \cup_{K^t} & & \cup_{H^t} \\ B_0 & \subset_L & B_1 \end{array}$$

is a symmetric commuting square of infinite dimensional multi-matrix algebras. By theorem 17.4 we then get a pair of hyperfinite II_1 -factors $A \subset B$ with

$$\dim\{A' \cap B\} \leq \left(\min\{ \text{1-norm of rows of } K^t \text{ and } H^t \} \right)^2.$$

This proves the assertion. □

18 Constructing Commuting Squares of Infinite Multi-matrix Algebras

The proof of the bi-unitary condition (1.9) for commuting squares of finite multi-matrix algebras, only uses local properties of the involved Bratteli diagrams. All the arguments on pp 1–9 proving theorem 1.7, may be repeated for inclusions of infinite dimensional multi-matrix algebras, provided that the inclusion matrices correspond to locally finite, countably infinite graphs. In particular the blocks, $u^{(i,k)}$ and $v^{(j,l)}$ of u and v are finite dimensional unitaries. Furthermore we may conclude

Theorem 18.1 *Let $G \ H \ K$ and L be adjacency matrices for locally finite, countably infinite, bipartite graphs, such that*

$$GH = KL \quad \text{and} \quad G^t K = H L^t.$$

Then the following conditions are equivalent

(a) *There exists a (symmetric) commuting square*

$$(A \subset B \subset D, \ A \subset C \subset D, \ tr_D)$$

of infinite dimensional multi-matrix algebras, with inclusion matrices

$$\begin{array}{ccc} C & \subset_L & D \\ \cup_K & & \cup_H \\ A & \subset_G & B. \end{array}$$

(b) *There exists a pair of matrices (u, v) satisfying the bi-unitary condition, i.e.*

$$u = \bigoplus_{(i,k)} u^{(i,k)}, \quad v = \bigoplus_{(j,l)} v^{(j,l)}$$

where the direct summands

$$u^{(i,k)} = \left(u_{(j,\rho,\sigma)(l,\phi,\psi)}^{(i,k)} \right)_{\substack{(i,j,k,\rho,\sigma) \in \mathcal{S} \\ (i,l,k,\phi,\psi) \in \mathcal{T}}},$$

$$v^{(j,l)} = \left(v_{(i,\rho,\phi)(k,\sigma,\psi)}^{(j,l)} \right)_{\substack{(i,j,k,\rho,\sigma) \in \mathcal{S} \\ (i,l,k,\phi,\psi) \in \mathcal{T}}}$$

are unitary matrices and

$$v_{(i,\rho,\phi)(k,\sigma,\psi)}^{(j,l)} = \sqrt{\frac{\alpha_i \delta_k}{\beta_j \gamma_l}} u_{(j,\rho,\sigma)(l,\phi,\psi)}^{(i,k)}$$

Here $\alpha_i, \beta_j, \gamma_l$ and δ_k are the trace weights on A, B, C resp. D coming from tr_D , and the indices $i, j, k, l, \rho, \sigma, \phi$ and ψ are as in theorem 1.7.

We can also conclude the statements of proposition 1.11 for infinite dimensional multi-matrix algebras, since also this proof is only concerned with local properties of the involved Bratteli diagrams.

Proposition 18.2 *If*

$$(A \subset_G B \subset_H D, A \subset_K C \subset_L D, tr_D)$$

is a symmetric commuting square of infinite dimensional multi-matrix algebras, such that the Bratteli diagrams $\Gamma_G, \Gamma_H, \Gamma_K$ and Γ_L are connected, locally finite, countably infinite graphs, then

- (I) $\|K\| = \|H\|$. Moreover tr_D is the Markov trace of the embedding $C \subset D$, and $tr_D|_B$ is the Markov trace of the embedding $A \subset B$.
- (II) $\|G\| = \|L\|$. Moreover tr_D is the Markov trace of the embedding $B \subset D$, and $tr_D|_C$ is the Markov trace of the embedding $A \subset C$.

Hence, if we, via the path model described in chapter I, construct a square of infinite dimensional multi-matrix algebras

$$\begin{array}{ccc} C & \subset_L & D \\ \cup_K & & \cup_H \\ A & \subset_G & B \end{array}$$

with

$$GH = KL \quad \text{and} \quad G^t K = HL^t,$$

where the inclusions are defined by locally finite, countably infinite, connected Bratteli diagrams, and have a trace defined on D by a summable vector, then, by proposition 18.2 and [S] theorem 6.4, this trace is the only trace for which there can exist a symmetric commuting square of infinite dimensional multi-matrix algebras, with the given Bratteli diagrams. To show the existence of a symmetric commuting square, with the the above inclusions, we need to show the existence of (u, v) , from theorem 18.1, satisfying the bi-unitary condition 1.9.

Remarks on Some Infinite Graphs

We will now be concerned with the properties of some well studied infinite graphs.

19 The Graphs Determined by Shearer have Summable Perron-Frobenius Vectors

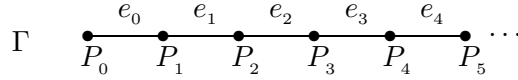
We will show that the eigenvector, corresponding to the largest eigenvalue of the infinite graphs constructed in [Sh] is summable.

If Γ is a graph we let $\lambda(\Gamma)$ denote the largest eigenvalue of the adjacency matrix, Δ_Γ , of Γ . In [Sh] it is proved that for any real number $\lambda > \sqrt{2 + \sqrt{5}}$ there exists a sequence of graphs $\Gamma_1^\lambda, \Gamma_2^\lambda, \dots$ such that

$$\lim_{n \rightarrow \infty} \lambda(\Gamma_n^\lambda) = \lambda.$$

Another way to view this construction is, that for $\lambda > \sqrt{2 + \sqrt{5}}$ there is a countably infinite graph, Γ_λ , with $\lambda(\Gamma_\lambda) = \lambda$. The description of Γ_λ is as follows.

For $\lambda > \sqrt{2 + \sqrt{5}}$ we choose $x > 0$ such that $\lambda = e^x + e^{-x}$, and consider the graph consisting of the vertices P_0, P_1, \dots and edges e_0, e_1, \dots where e_i connects P_i and P_{i+1} , i.e. the graph



The numbers $n_k, r_k, a_k, k \in \mathbb{N} \cup \{0\}$ are defined inductively by $n_0 = 0, r_1 = \lambda, a_0 = 1$ and

$$\begin{aligned} \text{(a)} \quad n_k &= \max\{j \in \mathbb{Z} \mid \lambda - \frac{1}{r_k} - \frac{j}{\lambda} \geq e^{-x}\}, \quad k \geq 1 \\ \text{(b)} \quad r_{k+1} &= \lambda - \frac{1}{r_k} - \frac{n_k}{\lambda}, \quad k \geq 1 \\ \text{(c)} \quad a_k &= r_k a_{k-1}, \quad k \geq 1 \end{aligned}$$

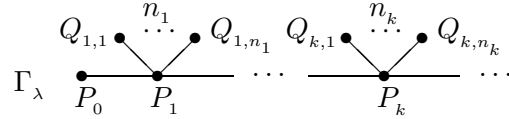
Proposition 19.1 *With the above notation the graph Γ_λ given by the graph Γ with n_k leaves, $Q_{k,1}, \dots, Q_{k,n_k}$, added at the vertex P_k , has norm of its adjacency matrix equal to λ . Furthermore the corresponding Perron-Frobenius vector ξ is given by the coordinates*

$$\begin{aligned} \xi(P_k) &= a_k \\ \xi(Q_{k,j}) &= \frac{a_k}{\lambda} \quad \text{if } n_k \neq 0 \end{aligned}$$

Proof From [Sh] it follows that the vector ξ , listed above, is an eigenvector for the adjacency matrix Δ_{Γ_λ} of Γ_λ corresponding to the eigenvalue λ . That the vector is also a Perron-Frobenius

vector follows since 1) Δ_{Γ_λ} is symmetric and irreducible, 2) $\xi(v) > 0$ for all vertices v , 3) the result Thm 6.4 from [S] and 4) our proof that the above vector is summable. \square

To sum up: For $\lambda > \sqrt{2 + \sqrt{5}} = e^x + e^{-x}$ we are looking at the graph



Lemma 19.2 *In the above notation $n_k \leq [\lambda^2] - 2 = n_\lambda$, where $[\lambda^2]$ denotes the integer part of λ^2 .*

Proof Let $\text{star}(n)$ denote the graph with vertices $0, 1, \dots, n$ and edges e_1, \dots, e_n , where e_i joins the vertices 0 and i . Then

$$\lambda(\text{star}(n)) = \sqrt{n}.$$

Hence Γ_λ can only contain $\text{star}(n)$ as a subgraph if $[\lambda^2] \geq n$, and the maximal number of leaves that can be added to a P_k is $[\lambda^2] - 2$. \square

Lemma 19.3

1. For $k \geq 2$ we have $e^{-x} \leq r_k \leq \frac{1}{\lambda} + e^{-x}$.
2. For $\lambda > \sqrt{2 + \sqrt{5}}$ we have $\frac{1}{\lambda} + e^{-x} < e^x$, and $\lambda = \sqrt{2 + \sqrt{5}}$ implies $\frac{1}{\lambda} + e^{-x} = e^x$.

Proof

1. From the property (a) we have

$$\lambda - \frac{1}{r_k} - \frac{n_k}{\lambda} \geq e^{-x} \text{ and } \lambda - \frac{1}{r_k} - \frac{n_k + 1}{\lambda} < e^{-x}.$$

So by property (b) we get

$$e^{-x} \leq r_{k+1} \leq \frac{1}{\lambda} + e^{-x}.$$

2. The solution to $\frac{1}{\lambda} + e^{-x} = e^x$ is $e^x = \sqrt{\frac{1}{2}(1 + \sqrt{5})}$, corresponding to

$$\lambda^2 = e^{2x} + 2 + e^{-2x} = 2 + \sqrt{5}.$$

This now implies the assertion. \square

Lemma 19.4 *On the interval $[e^{-x}, e^x]$ the iteration $t_{k+1} = \lambda - \frac{1}{t_k}$ is non-decreasing with fix points e^{-x} and e^x .*

Proof If \tilde{t} is a fix point for $t_k \mapsto \lambda - \frac{1}{t_{k-1}}$ we have

$$\tilde{t} = \frac{\lambda\tilde{t} - 1}{\tilde{t}} \Leftrightarrow \tilde{t} = \frac{\lambda \pm \sqrt{\lambda^2 - 4}}{2}.$$

Since $\lambda^2 - 4 = (e^x - e^{-x})^2$ we get that $\tilde{t} = e^x$ or $\tilde{t} = e^{-x}$.

Consider $x_{k+1} - x_k = \lambda - \frac{1}{x_k} - x_k$, then $x_{k+1} - x_k \geq 0$ is equivalent to

$$x_k^2 - \lambda x_k + 1 \leq 0 \Leftrightarrow \frac{1}{2}(\lambda - \sqrt{\lambda^2 - 4}) \leq x_k \leq \frac{1}{2}(\lambda + \sqrt{\lambda^2 - 4}),$$

and hence the iteration is non-decreasing for $e^{-x} \leq x_k \leq e^x$. □

Lemma 19.5 *Assume $\lambda > \sqrt{2 + \sqrt{5}}$. If $n_k = 0$ for $k \geq k_0$, then $r_k = e^{-x}$ for $k \geq k_0 + 1$.*

Proof By (b) we have that $r_{k+1} = \lambda - \frac{1}{r_k}$, $k \geq k_0$. If $r_{k_0+1} > e^{-x}$ lemma 19.4 gives

$$r_{k_0+1} \leq r_{k_0+2} \leq \dots,$$

so $r_k \nearrow r$, with r a fix point for the iteration in lemma 19.4. The fix points are e^{-x} and e^x , and since $r > e^{-x}$ we must have $r = e^x$.

Using lemma 19.3 we get

$$r \leq \sup_{k \geq k_0} r_k \leq \lambda - e^{-x} < e^x.$$

This is a contradiction, so $r_{k_0+1} = e^{-x}$, and we must have

$$r_k = e^{-x} \text{ for } k \geq k_0 + 1.$$

□

Lemma 19.6 *Assume $\lambda > \sqrt{2 + \sqrt{5}}$ then, if $2 \leq k < l$ are integers, such that $n_k \neq 0$ and $n_l \neq 0$ but $n_j = 0, k < j < l$, then there is a constant $\epsilon(\lambda)$ (independent of k, l) such that*

$$r_{k+1} r_{k+2} \cdots r_l \leq (1 - \epsilon(\lambda))^{\frac{l-k}{2}}.$$

Proof In the vertex P_k , ξ satisfies the equation

$$\begin{aligned} a_{k-1} + a_{k+1} + \frac{n_k}{\lambda} a_k &= \lambda a_k \\ \Downarrow \\ \frac{a_{k-1}}{a_k} + \frac{a_{k+1}}{a_k} + \frac{n_k}{\lambda} &= \lambda \\ \Downarrow \\ \frac{1}{r_k} + r_{k+1} + \frac{n_k}{\lambda} &= \lambda. \end{aligned}$$

Using lemma 19.3, we then have

$$r_{k+1} = \lambda - \frac{n_k}{\lambda} - \frac{1}{r_k} \leq \lambda - \frac{1}{\lambda} - \left(\frac{1}{\lambda} + e^{-x} \right)^{-1}.$$

We also have $r_l \leq \frac{1}{\lambda} + e^{-x}$, and so we get

$$\begin{aligned} r_{k+1} r_l &\leq \left(\lambda - \frac{1}{\lambda} - \left(\frac{1}{\lambda} + e^{-x} \right)^{-1} \right) \left(\frac{1}{\lambda} + e^{-x} \right) \\ &= \left(\lambda - \frac{1}{\lambda} \right) \left(\frac{1}{\lambda} + e^{-x} \right) - 1 \\ &= \frac{\lambda e^{-x} (\lambda^2 - 1) - 1}{\lambda^2}. \end{aligned}$$

The denominator can be rewritten as $e^{2x} + 1 - 2e^{-2x} - e^{-4x}$, using $\lambda = e^x + e^{-x}$, and the numerator is $e^{2x} + 2 + e^{-2x}$.

We want to show that $r_{k+1} r_l < 1$, which is equivalent to

$$e^{2x} + 1 - 2e^{-2x} - e^{-4x} < e^{2x} + 2 + e^{-2x} \Leftrightarrow \lambda > \sqrt{2 + \sqrt{5}}.$$

Hence we can find $\epsilon(\lambda) = 1 - \frac{\lambda e^{-x} (\lambda^2 - 1) - 1}{\lambda^2} > 0$ such that $r_{k+1} r_l < 1 - \epsilon(\lambda)$.

We will show that $r_{k+2} r_{l-1} \leq 1 - \epsilon(\lambda)$ for $l - k \geq 3$.

Since n_{k+1}, n_{k+2} and n_{l-1} all equal 0, (b) yields that $r_{k+2} = \lambda - \frac{1}{r_{k+1}}$ and $r_{l-1} = \frac{1}{\lambda - r_l}$, so

$$r_{k+2} r_{l-1} = \left(\lambda - \frac{1}{r_{k+1}} \right) \left(\frac{1}{\lambda - r_l} \right).$$

It suffices to show $r_{k+2} r_{l-1} \leq r_{k+1} r_l$, which is equivalent to showing

$$\frac{1}{r_{k+1}} \left(\lambda - \frac{1}{r_{k+1}} \right) \leq r_l (\lambda - r_l).$$

For $j \in \{k+1, \dots, l-1\}$ we have $r_j = \lambda - \frac{1}{r_{j-1}}$, so by lemma 19.4 r_{k+1}, \dots, r_l is a non-decreasing sequence.

Since ξ is eigenvector we have $\frac{1}{r_{k+1}} + r_{k+2} = \lambda$, and hence

$$\frac{1}{r_{k+1}} = \lambda - r_{k+2} \geq \lambda - r_l.$$

We also have $\frac{1}{r_{k+1}} \geq r_l$, since $r_{k+1} r_l < 1$, and we may conclude

$$\frac{1}{r_{k+1}} \geq \max\{r_l, \lambda - r_l\}.$$

By the properties of the function $t \mapsto t(\lambda - t), 0 \leq t \leq \lambda$ we now conclude

$$\frac{1}{r_{k+1}} \left(\lambda - \frac{1}{r_{k+1}} \right) \leq r_l (\lambda - r_l)$$

and hence

$$r_{k+2}r_{l-1} \leq 1 - \epsilon(\lambda).$$

Continuing this argument we have that all the numbers $r_{k+1}r_l, r_{k+2}r_{l-1}, \dots, r_{k+t}r_{l-t+1}$ are dominated by $1 - \epsilon(\lambda)$ provided $k+t \leq l-t+1$.

For $l-k$ even we have $r_{k+1} \cdots r_l \leq (1 - \epsilon(\lambda))^{\frac{l-k}{2}}$.

For $l-k$ odd, $l-k = 2n+1$, we have

$$r_{k+1} \cdots r_{k+n} r_{k+n+2} \cdots r_l \leq (1 - \epsilon(\lambda))^{\frac{l-k-1}{2}}$$

and since the previous argument gives $r_{k+n+1} = \sqrt{r_{k+n+1}r_{k+n+1}} \leq \sqrt{1 - \epsilon(\lambda)}$ we have

$$r_{k+1} \cdots r_{k+n} r_{k+n+1} r_{k+n+2} \cdots r_l \leq (1 - \epsilon(\lambda))^{\frac{l-k}{2}}.$$

□

Proposition 19.7 For $\lambda > \sqrt{2 + \sqrt{5}}$ the eigenvector ξ of Γ_λ given by

$$\begin{aligned} \xi(P_k) &= a_k \\ \xi(Q_{k,j}) &= \frac{a_k}{\lambda} \quad \text{if } n_k \neq 0 \end{aligned}$$

is summable.

Proof If $n_k = 0$ for $k \geq k_0$, lemma 19.5 implies that $r_k = e^{-x}$ for $k \geq k_0 + 1$. By definition $a_{k+1} = r_{k+1}a_k$, so we have

$$a_{k_0+n} = e^{-nx}a_{k_0},$$

and the eigenvector ξ is seen to be summable.

If n_k does not eventually equal 0, we consider $k < l$ such that $n_k \neq 0$ and $n_l \neq 0$ but $n_j = 0$ for $k < j < l$.

By lemma 19.6 we have

$$a_l \leq (1 - \epsilon(\lambda))^{\frac{l-k}{2}} a_k.$$

If we consider $\log a_k, \log a_{k+1}, \dots, \log a_l$ this is a convex function of the index, since the differences $\log r_{k+1}, \log r_{k+2}, \dots, \log r_l$ satisfy

$$\log r_{k+1} \leq \log r_{k+2} \leq \dots \leq \log r_l$$

and it follows that

$$a_n \leq (1 - \epsilon(\lambda))^{\frac{n-k}{2}} a_k, \quad k \leq n \leq l.$$

If we let $k_0 \geq 2$ denote the smallest integer such that $n_{k_0} \neq 0$ and note that $\epsilon(\lambda)$ is independent of k_0 we get

$$a_n \leq (1 - \epsilon(\lambda))^{\frac{n-k_0}{2}} a_{k_0}, \quad n \geq k_0.$$

Using lemma 19.2 we now get

$$\begin{aligned}
\|\xi\|_1 &= \sum_{j < k_0} a_j + \sum_{j \geq k_0} (a_j + n_j \frac{a_j}{\lambda}) \\
&\leq \sum_{j < k_0} a_j + (\frac{n_{k_0}}{\lambda} + 1) \sum_{j \geq k_0} a_j \\
&\leq \sum_{j < k_0} a_j + (\frac{n_{k_0}}{\lambda} + 1) a_{k_0} \sum_{j \geq k_0} (1 - \epsilon(\lambda))^{\frac{j - n_{k_0}}{2}} \\
&< \infty.
\end{aligned}$$

□

20 Not All Shearer-Graphs Can Define Commuting Squares

If we try to build a commuting square of infinite dimensional multi-matrix algebras, with one of the infinite graphs, Γ_λ , defined by Shearer (see section 19) as the index defining side, there are not many obvious choices of the form of the inclusion matrices that define the other sides of the commuting square. The other inclusions have to have compatible Perron-Frobenius eigenvector, so a polynomial applied to the adjacency matrix of Γ_λ is a possibility (as described below), and it does not seem likely that one can define any other form of inclusions, that will work in general, to construct such commuting squares. We will show, that if Γ_λ is the index defining inclusion of a commuting square of infinite multi-matrix algebras of the above form, then Γ_λ has to be eventually periodic.

As a consequence Shearer's result, that the set of Perron-Frobenius eigenvalues of infinite graphs contains all of $\mathcal{F} = \{x \in \mathbb{R} | x \geq \sqrt{2 + \sqrt{5}}\}$, cannot be used to produce values of the index of irreducible Hyperfinite II_1 -factors which form a closed subset of \mathcal{F} .

The argument is as follows.

Let $\lambda > \sqrt{2 + \sqrt{5}} = e^x + e^{-x}$ and let Γ_λ be the graph discussed in (19). Let $\Delta_{\lambda, bp}$ be the adjacency matrix of a bi-partition of Γ_λ . I.e. $\Delta_{\lambda, bp} = \begin{pmatrix} 0 & G \\ G^t & 0 \end{pmatrix}$. Let p be a polynomial such that $p(\Delta_{\lambda, bp})$ corresponds to the adjacency matrix of a bi-partite graph. If the exponents of p are all even, $p(\Delta_{\lambda, bp})$ is of the form $\begin{pmatrix} H & 0 \\ 0 & K \end{pmatrix}$. If the exponents are all odd, $p(\Delta_{\lambda, bp})$ is of the form $\begin{pmatrix} 0 & H \\ K & 0 \end{pmatrix}$.

Let n be the highest degree in $p(t)$ and c_n be the coefficient of t^n . Assume that there exists a commuting square of infinite dimensional multimatrix algebras of the form

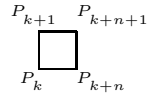
$$\begin{array}{ccc} C & \subset_K & D \\ \cup_G & & \cup_G \text{ if } n \text{ is even,} \\ A & \subset_H & B \end{array} \quad \begin{array}{ccc} C & \subset_K & D \\ \cup_G & & \cup_{G^t} \text{ if } n \text{ is odd.} \\ A & \subset_H & B \end{array}$$

If we look at the cycles involving the vertices P_k, P_{k+1}, P_{n+k} and P_{n+k+1}

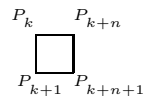


Then either

there is only one $c_n \times c_n$ unitary block, A , in u , corresponding to cycles of the form



or there is only one $c_n \times c_n$ unitary block, B , in v , corresponding to cycles of the form



In the first case, the scalar involved in the transition from u to v is $\sqrt{\frac{\xi(P_k)\xi(P_{k+n+1})}{\xi(P_{k+1})\xi(P_{k+n})}}$, and in

the second case, the scalar involved in the transition from v to u is $\sqrt{\frac{\xi(P_k)\xi(P_{k+n+1})}{\xi(P_{k+1})\xi(P_{k+n})}}$. So in either case we must have

$$\nu = \sqrt{\frac{\xi(P_k)\xi(P_{k+n+1})}{\xi(P_{k+1})\xi(P_{k+n})}} \leq 1 \quad (20.1)$$

since either νA must be part of a unitary in v , or νB must be part of a unitary in u . In other terms (20.1) can be stated as

$$r_{k+n} \leq r_k,$$

and consequently we must have

$$r_k \geq r_{k+n} \geq r_{k+2n} \geq \cdots \geq e^{-x}, \quad (20.2)$$

where the last inequality comes from lemma 19.3. Hence we have

$$\lim_{j \rightarrow \infty} r_{k+nj} = r \geq e^{-x}.$$

By definition n_k is the largest integer such that

$$\frac{1}{r_k} + \frac{n_k}{\lambda} + e^{-x} \leq \lambda,$$

so (20.2) implies

$$n_k \geq n_{k+n} \geq n_{k+2n} \geq \cdots \geq 0,$$

and since they are all integers, there exists j_k such that n_{k+jn} is constant for $j \geq j_k$.

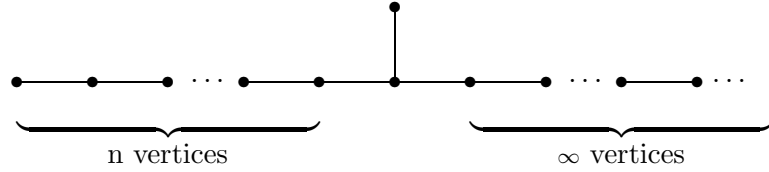
Using this argument for $k = 1, 2, \dots, n$ we find J such that $n_i = n_{i+n}$, for all $i \geq J$, i.e. Γ_λ is eventually periodic.

Some Index Values Which Do Not Occur From Finite Graphs

In this chapter we will show that the largest eigenvalue, λ_n , of the graphs $T(1, n, \infty)$ cannot occur as eigenvalues of finite graphs. Furthermore we will construct commuting squares which will give λ_2^2, λ_3^2 and λ_4^2 as index for a pair of irreducible Hyperfinite II_1 -factors.

21 The Largest Eigenvalue of $T(1, n, \infty)$ Does Not Occur as Eigenvalue of a Finite Graph

We will look at the graph $T(1, n, \infty)$ as defined by Hoffmann in [Hof].



If we let λ denote the largest eigenvalue of this graph, then $\lambda = e^x + e^{-x}$ for some $x > 0$. If we put $\rho = e^{2x}$, ρ satisfies the equation (see [Hof])

$$\rho^{n+2} - \rho^{n+1} - \rho^n + 1 = 0, \quad (21.1)$$

or, if we divide by $\rho - 1$

$$\rho^{n+1} - \rho^{n-1} - \rho^{n-2} - \dots - \rho - 1. \quad (21.2)$$

In this section we will show that $\lambda(T(1, n, \infty))$ is an algebraic integer, and that $\lambda(T(1, n, \infty))$ does not occur as eigenvalue for any finite graph.

Remark 21.1 If we for $\lambda = e^x + e^{-x}$ define

$$P_n(\lambda) = \frac{e^{(n+1)x} - e^{-(n+1)x}}{e^x - e^{-x}}$$

then

$$\begin{aligned} P_0(\lambda) &= 1 \\ P_1(\lambda) &= \frac{e^{2x} - e^{-2x}}{e^x - e^{-x}} = \lambda. \end{aligned}$$

Also

$$(e^x + e^{-x}) \frac{e^{(n+1)x} - e^{-(n+1)x}}{e^x - e^{-x}} - \frac{e^{nx} - e^{-nx}}{e^x - e^{-x}} = \frac{e^{(n+2)x} - e^{-(n+2)x}}{e^x - e^{-x}},$$

so $P_n(\lambda) = R_n(\lambda)$. If we expand $P_n(\lambda)$ as a finite sum of quotients, we have

$$R_n(\lambda) = e^{nx} + e^{(n-2)x} + e^{(n-4)x} + \dots + e^{-nx}.$$

Proposition 21.2 *The largest eigenvalue, $\lambda_n = \lambda(T(1, n, \infty))$, for $T(1, n, \infty)$ is a root of the $(2n + 2)$ -degree polynomial*

$$K_n(\lambda) = (R_{n+3}(\lambda) - R_{n+1}(\lambda) - R_{n-1}(\lambda))R_{n-1}(\lambda) - 1.$$

Proof Let $\lambda_0 = e^{x_0} + e^{-x_0}$ for some $x_0 > 0$ and let $\rho_0 = e^{2x_0}$. Then

$$\phi_n(\rho_0) = \rho_0^{n+1} - \rho_0^{n-1} - \rho_0^{n-2} - \dots - \rho_0 - 1 = 0.$$

Hence also

$$-\phi_n(\rho_0)\phi_n(\rho_0^{-1}) = 0.$$

Dividing the first factor by $e^{(n-1)x_0}$ and multiplying the second by $e^{(n-1)x_0}$ we get

$$\begin{aligned} 0 &= -(e^{(n+3)x_0} - e^{(n-1)x_0} - e^{(n-3)x_0} - \dots - e^{-(n-1)x_0}) \\ &\quad \cdot (e^{-(n+3)x_0} - e^{-(n-1)x_0} - e^{-(n-3)x_0} - \dots - e^{(n-1)x_0}) \\ &= -(e^{(n+3)x_0} - R_{n-1}(\lambda_0)) (e^{-(n+3)x_0} - R_{n-1}(\lambda_0)) \\ &= (e^{(n+3)x_0} + e^{-(n+3)x_0}) R_{n-1}(\lambda_0) - R_{n-1}(\lambda_0)^2 - 1 \\ &= (R_{n+3}(\lambda_0) - R_{n+1}(\lambda_0) - R_{n-1}(\lambda_0)) R_{n-1}(\lambda_0) - 1 \\ &= K_n(\lambda_0), \end{aligned}$$

where the second equality follows by remark 21.1. □

Proposition 21.3 *Consider the polynomial*

$$K_n(\lambda) = (R_{n+3}(\lambda) - R_{n+1}(\lambda) - R_{n-1}(\lambda)) R_{n-1}(\lambda) - 1, \quad \lambda \in \mathbb{C}.$$

Then

1. *For n even the only real roots of $K_n(\lambda)$ are $\pm\lambda(T(1, n, \infty))$.*
2. *For n odd the only real roots of $K_n(\lambda)$ are $\pm\lambda(T(1, n, \infty))$ and 0.*

Proof Since K_n is an even polynomial, it is enough to consider $\lambda \geq 0$.

I. If $\lambda > 2$, we may write $\lambda = e^x + e^{-x}$, $x \geq 0$ and put $\rho = e^{2x} > 1$. Then

$$K_n(\lambda) = -\phi_n(\rho)\phi_n(\rho^{-1})$$

where

$$\phi_n(\rho) = \rho^{n+1} - \rho^{n-1} - \rho^{n-2} - \dots - \rho - 1.$$

Since a) $\rho \mapsto \frac{\phi_n(\rho)}{\rho^{n+1}}$ is strictly increasing on \mathbb{R}_+ and b) $\phi_n(0) = -1$, $\lim_{\rho \rightarrow \infty} \phi_n(\rho) = \infty$, the equation $\phi_n(\rho) = 0$ has precisely one solution, ρ_0 , in \mathbb{R}_+ . Moreover $\phi_n(1) = 1 - n < 0$, so the solution ρ_0 is in the interval $1 < \rho_0 < \infty$. Hence the equation $K_n(\lambda) = 0$ has exactly one solution, λ_0 , with $2 < \lambda_0 < \infty$. This value must then equal $\lambda(T(1, n, \infty))$.

II. If $0 \leq \lambda \leq 2$, we can write $\lambda = 2 \cos \theta$, $0 \leq \theta \leq \frac{\pi}{2}$. Put $\rho = e^{i\theta}$. Then

$$K_n(\lambda) = \left| \rho^{n+1} - \rho^{n-1} - \rho^{n-2} - \dots - \rho - 1 \right|^2,$$

so $K_n(\lambda) = 0$ implies

$$\rho^{n+1} - \rho^{n-1} - \rho^{n-2} - \dots - \rho - 1 = 0. \quad (21.3)$$

This is in fact (21.2), which implies (21.1)

$$\rho^{n+2} - \rho^{n+1} - \rho^n + 1 = 0,$$

or

$$\rho^2 - \rho - 1 = -\rho^{-n}.$$

Hence $|\rho^2 - \rho - 1| = 1$. Since $\rho = e^{i\theta}$ we have

$$|\rho^2 - \rho - 1| = 3 - 2\operatorname{Re}(\rho) - 2\operatorname{Re}(\rho^2) + 2\operatorname{Re}(\rho) = 3 - 2 \cos 2\theta = 1 + 4 \sin^2 \theta,$$

i.e. $|\rho^2 - \rho - 1| > 1$ except for $\theta = p\pi$, $p \in \mathbb{Z}$, or equivalently: $|\rho^2 - \rho - 1| > 1$ except for $\rho = \pm 1$. The case $\rho = 1$ is excluded by (21.3), and $\rho = -1$ is a solution to (21.3) if and only if n is odd. Since $\rho = -1$ corresponds to $\lambda = 0$, we have proved the assertions of the proposition. \square

Theorem 21.4 *The numbers $\lambda_n = \lambda(T(1, n, \infty))$, $n \geq 2$ and $\lambda_\infty = \lambda(T(1, \infty, \infty))$ are algebraic integers, and none of these numbers can be obtained as an eigenvalue of a finite graph.*

Proof It is easy to check that $\lambda_\infty = \sqrt{2 + \sqrt{5}}$ is a root in the polynomial $Q(\lambda) = \lambda^4 - 4\lambda^2 - 1$.

Q is irreducible since: The roots of Q are $\pm\sqrt{2 + \sqrt{5}}$ and $\pm i\sqrt{\sqrt{5} - 2}$. None of these roots are integers, so any irreducible factor of Q is of degree at least 2. Moreover the two complex conjugate roots must be roots of the same irreducible factor of Q . Hence the only possible factorization of Q into monic, irreducible polynomials, would be $Q = Q_1 Q_2$ where

$$Q_1(\lambda) = (\lambda - \sqrt{2 + \sqrt{5}})(\lambda + \sqrt{2 + \sqrt{5}}) = \lambda^2 - 2 - \sqrt{5}$$

and

$$Q_2(\lambda) = (\lambda - i\sqrt{\sqrt{5} - 2})(\lambda + i\sqrt{\sqrt{5} - 2}) = \lambda^2 - 2 + \sqrt{5}.$$

However Q_1 and Q_2 do not have integer coefficients, and hence Q is irreducible.

Assume that λ_∞ is an eigenvalue of the adjacency matrix Δ_Γ of the finite graph Γ . The characteristic polynomial

$$f(\lambda) = \det(\lambda I - \Delta_\Gamma)$$

is monic with integer coefficients. Moreover, since Δ_Γ is symmetric, all the roots of f are real. Since $f(\lambda_\infty) = 0 = Q(\lambda_\infty)$, the irreducibility of Q implies that Q divides f . This is impossible because Q has non-real roots.

We will now turn to $\lambda_n = \lambda(T(1, n, \infty))$. By proposition 21.2, λ_n is root in a monic polynomial, K_n , with integer coefficients. Let Q_n be the minimal monic polynomial over \mathbb{Q} , which has λ_n as a root. Since λ_n is an algebraic integer, Q_n has integer coefficients (see [ST] lemma 2.12).

We claim that Q_n must have non-real roots. Indeed, since Q_n is a factor in K_n , the only possible real roots of Q_n are $\pm\lambda_n$ and 0 by proposition 21.3. However 0 is not a root of Q_n , because Q_n is irreducible. Hence, if Q_n has only real roots, it must be of the form

$$Q_n(\lambda) = \lambda - \lambda_n$$

or

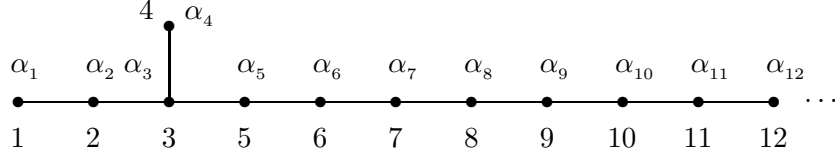
$$Q_n(\lambda) = (\lambda - \lambda_n)(\lambda + \lambda_n) = \lambda^2 - \lambda_n^2.$$

Here it is used that irreducible polynomials do not have multiple roots. (See [ST] corollary 1.2) However, $\lambda_n \notin \mathbb{Z}$ and $\lambda_n^2 \notin \mathbb{Z}$, because $2 < \lambda_n < \sqrt{2 + \sqrt{5}} < \sqrt{5}$, which is a contradiction. Hence Q_n has at least one non-real root, and, as in the case λ_∞ , it follows that λ_n is not an eigenvalue of a finite graph. \square

Remark 21.5 The idea to the above proof is due to P. de la Harpe, [PH], who used the method to prove that λ_2, λ_3 and λ_∞ are not eigenvalues of any finite graph.

22 A Commuting Square Based on $T(1, 2, \infty)$

We will look at the graph $T(1, 2, \infty)$ as defined by Hoffmann in [Hof].



If $\lambda = e^x + e^{-x}$ is the Perron-Frobenius eigenvalue of $T(1, 2, \infty)$, and ρ denotes e^{2x} then ρ satisfies the equation (see [Hof])

$$\rho^3 - \rho - 1 = 0 \quad (22.1)$$

corresponding to $\rho \cong 1.32472$ and $\lambda \cong 2.01980$.

The corresponding coordinates of the Perron-Frobenius vector, α , are

$$\begin{aligned} \alpha_1 &= e^{-8x} & \alpha_2 &= e^{-3x} \\ \alpha_3 &= 1 & \alpha_4 &= e^{-5x} \end{aligned}$$

and for $i > 5$ we have $\alpha_i = e^{4-i}$.

These are determined as follows. α_4 must equal $\frac{1}{\lambda}$, when we have scaled the vector to 1 at the vertex 3. If we use the equation (22.1) this is easily seen to equal e^{-5x} . If z denotes α_2 then $z = \lambda - \frac{1}{\lambda} - e^{-x} = \frac{e^{2x}}{\lambda}$, and we can use (22.1) to show that this equals e^{-3x} . Finally $\alpha_1 = \frac{\alpha_2}{\lambda} = e^{-8x}$.

We will construct a commuting square with the adjacency matrix, Δ , of $T(1, 2, \infty)$ as the index defining inclusion. More precisely we let $\Delta_{bp} = \begin{pmatrix} 0 & H \\ K & 0 \end{pmatrix}$ be the adjacency matrix of a bi-partition of $T(1, 2, \infty)$. We will construct a commuting square of the form

$$\begin{array}{ccc} C & \subset_L & D \\ \cup_K & & \cup_H \\ A & \subset_G & B \end{array}$$

where G and L are defined by evaluating some polynomial P in Δ , and then apply the bi-partition used to obtain Δ_{bp} to the graph corresponding to $P(\Delta)$. I.e. the exponents in P must all be even or all be odd. If all the exponents are odd, H and L are found as $\begin{pmatrix} 0 & G \\ L & 0 \end{pmatrix}$, and if all the exponents are even G and L can be found as $\begin{pmatrix} G & 0 \\ 0 & L \end{pmatrix}$.

In the case of the A_n -graphs (see chapter 8) we used the polynomials R_k , defined inductively by

$$\begin{aligned} R_0(t) &= 1 \\ R_1(t) &= t \\ R_{k+1}(t) &= tR_k(t) - R_{k-1}(t). \end{aligned}$$

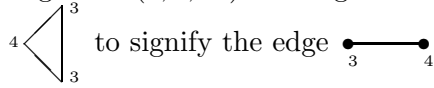
These polynomials occurred naturally as coordinates of the Perron-Frobenius vector of A_n . In the present case the R_n 's evaluated in Δ will also be positive (see [HW]), but because of the infinite “ray” we can find a polynomial, P , with “smaller” entries in $P(\Delta)$ which will work. Define the polynomials S_k by

$$\begin{aligned} S_1(t) &= t \\ S_2(t) &= t^2 - 2 \\ S_{k+1}(t) &= tS_k(t) - S_{k-1}(t). \end{aligned} \tag{22.2}$$

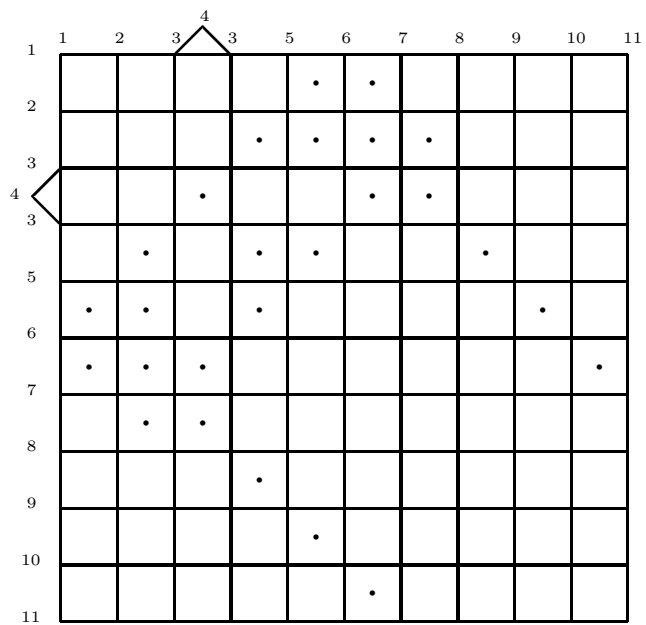
Then $S_4(\Delta)$ is positive, and given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Just like in the case of the A_n -graphs we can picture the matrices u and v in a diagram with the straight line, we introduce the notation edges of $T(1, 2, \infty)$ defining the “boxes” in the diagram. Since the graph is not a



We then have the following “boxes” which correspond to entries in u and v .



The south-east sloping rows of boxes, starting in the boxes corresponding to the cycles $\begin{smallmatrix} 5 & 3 \\ 8 & 9 \end{smallmatrix}$ respectively $\begin{smallmatrix} 9 & 8 \\ 3 & 5 \end{smallmatrix}$, all correspond to 1×1 blocks of both u and v .

The scalars defining the transition from u to v are all equal to 1 for these cycles, so a solution of u and v in these boxes is given by putting all the scalars equal to 1.

Using the coordinates of the Perron-Frobenius vector and reducing with the polynomial (22.1), we get the following table of the non-trivial values of the scalars defining the transition from u to v .

The scalars defining the transition from u to v .

	1	2	3	4	3	5	6	7	8	9	10
1						e^{-2x}	e^{-3x}				
2					e^x	e^{2x}	e^x	e^{2x}			
3							e^{2x}	e^{3x}			
4			1								
5		e^{-x}			1	e^x			1		
6	e^{2x}	e^{-2x}			e^{-x}						1
7	e^{3x}	e^{-x}	e^{-2x}								
8		e^{-2x}	e^{-3x}								
9					1						
10						1					

The moduli and of the elements in u and v can all be determined using the block structure of u and v , and the scalars in the above table. I.e. using that the moduli squared of the entries in a 2×2 unitary must be of the form $\begin{pmatrix} \alpha & 1-\alpha \\ 1-\alpha & \alpha \end{pmatrix}$, and that a 1×1 unitary is a complex scalar of length 1. The block structure of u and v is indicated by the thick lines in the following diagrams. We have solutions to u and v looking like

The entries of u .

	1	2	3	4	3	5	6	7	8	9	10
1						1	1				
2					e^{-x}	e^{-5x}	$-e^{-3x}$	e^{-2x}			
3							e^{-2x}	e^{-3x}			
4			1								
5		1			$-e^{-5x}$	e^{-x}			1		
6	e^{-2x}	e^{-3x}			1						1
7	e^{-3x}	$-e^{-2x}$	1								
8		1	1								
9					1						
10						1					

The entries of v .

	1	2	3	4	3	5	6	7	8	9	10
1						e^{-2x}	e^{-3x}				
2					1	e^{-3x}	$-e^{-2x}$	1			
3											
4			1				1	1			
5		e^{-x}			$-e^{-5x}$	1			1		
6	1	e^{-5x}			e^{-x}						1
7	1	$-e^{-3x}$	e^{-2x}								
8		e^{-2x}	e^{-3x}								
9					1						
10						1					

All that is left to show, is that the determined 2×2 matrices correspond to doubly stochastic matrices, i.e. that

$$(1) \quad (e^{-3x})^2 + (e^{-2x})^2 = 1$$

$$(2) \quad (e^{-5x})^2 + (e^{-1x})^2 = 1$$

(1) follows from (22.1) since

$$(e^{-3x})^2 + (e^{-2x})^2 = \rho^{-3} + \rho^{-1}.$$

(2) follows since

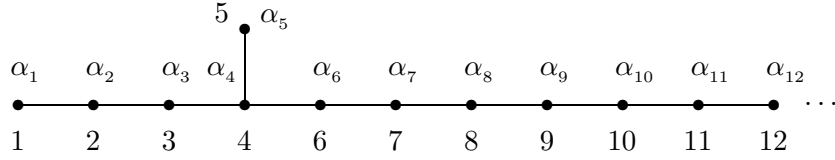
$$(e^{-5x})^2 + (e^{-1x})^2 = 1 \Leftrightarrow \rho^5 - \rho^4 - 1 = 0,$$

and

$$\rho^5 - \rho^4 - 1 = (\rho^2 - \rho + 1)(\rho^3 - \rho - 1) = 0.$$

23 A Commuting Square Based on $T(1, 3, \infty)$

We will look at the graph $T(1, 3, \infty)$ as defined by Hoffmann in [Hof].



If $\lambda = e^x + e^{-x}$ is the Perron-Frobenius eigenvalue of $T(1, 3, \infty)$, and ρ denotes e^{2x} then ρ satisfies the equation (see [Hof])

$$\rho^3 - \rho^2 - 1 = 0 \tag{23.1}$$

corresponding to $\rho \cong 1.46557$ and $\lambda \cong 2.03664$.

The corresponding coordinates of the Perron-Frobenius vector, α , are

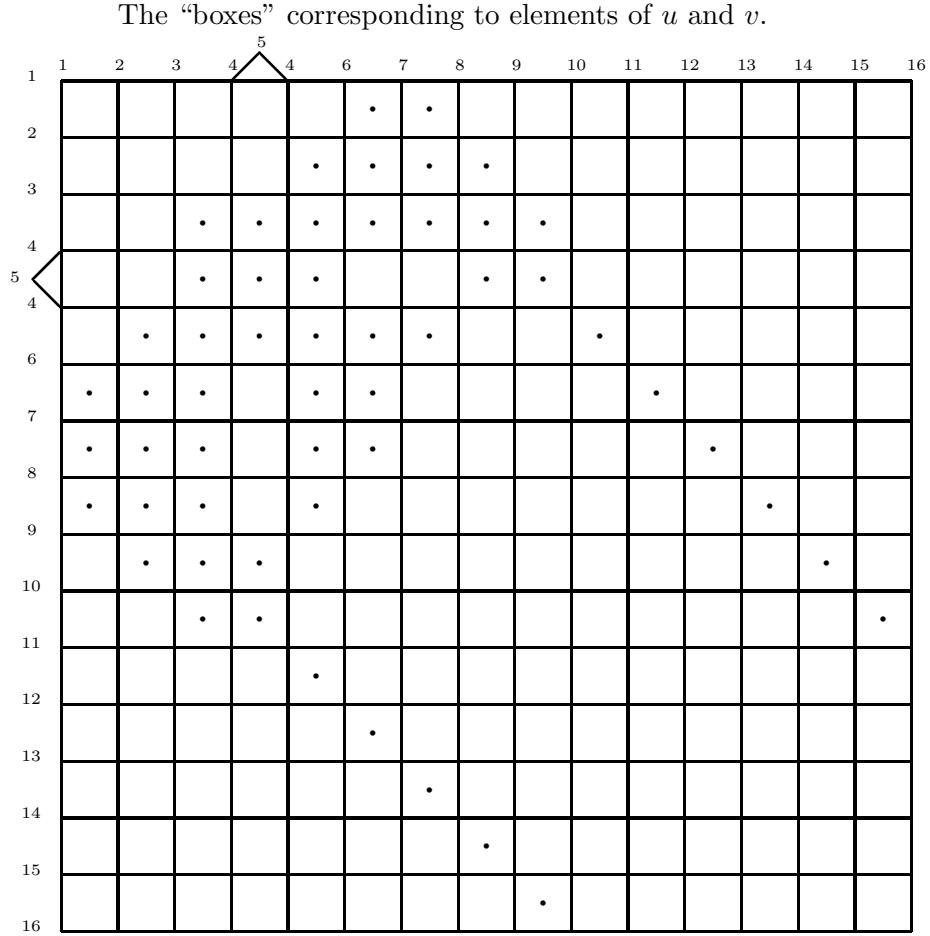
$$\begin{aligned} \alpha_1 &= \frac{e^{-4x}}{\lambda} & \alpha_2 &= e^{-4x} \\ \alpha_3 &= \frac{e^{2x}}{\lambda} & \alpha_4 &= 1 \\ \alpha_5 &= \frac{1}{\lambda} \end{aligned}$$

and for $i > 6$ we have $\alpha_i = e^{5-i}$.

We will construct a commuting square where the adjacency matrix, Δ , of $T(1, 3, \infty)$ gives the index defining inclusion, as described on page 153. As in the example $T(1, 2, \infty)$ we will look at the polynomials $S_n(t)$ defined in (22.2). In this case the polynomial $S_5(\Delta)$ is positive, with $S_5(\Delta)$ given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We then have the following “boxes” which correspond to entries in u and v .



The south-east sloping rows of boxes, starting in the boxes corresponding to the cycles $\begin{smallmatrix} 11 & 10 \\ 6 & 4 \end{smallmatrix}$ respectively

$\begin{smallmatrix} 4 & 6 \\ 10 & 11 \end{smallmatrix}$, all correspond to 1×1 blocks of both u and v .

The scalars defining the transition from u to v are all equal to 1 for these cycles, so a solution of u and v in these boxes is given by putting all the scalars equal to 1.

Using the coordinates of the Perron-Frobenius vector and reducing with the polynomial (23.1), we get the following table of the non-trivial values of the scalars defining the transition from u to v .

The scalars defining the transition from u to v .

				5							
	1	2	3	4	4	6	7	8	9	10	11
1						$\sqrt{\lambda^{-1}e^x}$	$\sqrt{\lambda^{-1}e^{-x}}$				
2					$\sqrt{\lambda^{-1}e^{5x}}$	$\sqrt{\lambda^{-1}e^{7x}}$	$\sqrt{\lambda^{-1}e^{5x}}$	$\sqrt{\lambda^{-1}e^{7x}}$			
3			$\lambda^{-1}e^{2x}$	$\lambda^{-1}e^x$	$\sqrt{\lambda^{-1}e^x}$	$\sqrt{\lambda^{-1}e^{3x}}$	$\sqrt{\lambda^{-1}e^x}$	$\sqrt{\lambda^{-1}e^{3x}}$	$\sqrt{\lambda^{-1}e^x}$		
4			$\lambda^{-1}e^x$	$\sqrt{\lambda^{-1}}$	$\sqrt{\lambda^{-1}e^{-x}}$			$\sqrt{\lambda^{-1}e^x}$	$\sqrt{\lambda^{-1}e^x}$		
5											
4		$\sqrt{\lambda^{-1}e^{5x}}$	$\sqrt{\lambda^{-1}e^x}$	$\sqrt{\lambda^{-1}e^{-x}}$	e^{-x}	1	e^{-x}			1	
6	$\sqrt{\lambda^{-1}e^x}$	$\sqrt{\lambda^{-1}e^{7x}}$	$\sqrt{\lambda^{-1}e^{3x}}$		1	e^x					
7	$\sqrt{\lambda^{-1}e^{-x}}$	$\sqrt{\lambda^{-1}e^{5x}}$	$\sqrt{\lambda^{-1}e^x}$		e^{-x}						
8		$\sqrt{\lambda^{-1}e^{7x}}$	$\sqrt{\lambda^{-1}e^{3x}}$	$\sqrt{\lambda^{-1}e^x}$							
9			$\sqrt{\lambda^{-1}e^x}$	$\sqrt{\lambda^{-1}e^{-x}}$							
10					1						
11											

The moduli and of the elements in u and v can all be determined using the block structure of u and v , and the scalars in the above table. (Just as in the example with $T(1, 2, \infty)$.) The block structure of u and v is indicated by the thick lines in the following diagrams. We have solutions to u and v

looking like

The entries of u

				5							
1	2	3	4	4	6	7	8	9	10	11	
1						1	1				
2					$\sqrt{\lambda e^{-5x}}$	e^{-4x}	$-e^{-2x}$	$\sqrt{\lambda e^{-7x}}$			
3			ξ	η	e^{-4x}	$-\sqrt{\lambda e^{-5x}}$	$\sqrt{\lambda e^{-7x}}$	e^{-2x}	1		
4			μ	ν	1			1	-1		
4											
5		$\sqrt{\lambda e^{-5x}}$	e^{-4x}	1	e^{-x}	e^{-3x}	1			1	
6	-1	e^{-4x}	$-\sqrt{\lambda e^{-5x}}$		e^{-3x}	$-e^{-x}$					
7	1	e^{-2x}	$\sqrt{\lambda e^{-7x}}$		1						
8		$\sqrt{\lambda e^{-7x}}$	$-e^{-2x}$	1							
9			1	1							
10					1						
11											

The entries of v .

				5							
	1	2	3	4	4	6	7	8	9	10	11
1						$\sqrt{\lambda^{-1}e^x}$	$\sqrt{\lambda^{-1}e^{-x}}$				
2											
3					1	$\sqrt{\lambda^{-1}e^x}$	$-\sqrt{\lambda^{-1}e^{-x}}$	1			
4			$\lambda^{-1}e^{2x}\xi$	$\lambda^{-1}e^x\eta$	$\sqrt{\lambda^{-1}e^{-7x}}$	$-e^{-x}$	e^{-3x}	$\sqrt{\lambda^{-1}e^{-x}}$	$\sqrt{\lambda^{-1}e^x}$		
5			$\lambda^{-1}e^x\mu$	$\lambda^{-1}\nu$	$\sqrt{\lambda^{-1}e^{-x}}$			$\sqrt{\lambda^{-1}e^x}$	$-\sqrt{\lambda^{-1}e^{-x}}$		
4											
6		1	$\sqrt{\lambda^{-1}e^{-7x}}$	$\sqrt{\lambda^{-1}e^{-x}}$	e^{-2x}	e^{-3x}	e^{-x}			1	
7	$-\sqrt{\lambda^{-1}e^x}$	$\sqrt{\lambda^{-1}e^{-x}}$	$-e^{-x}$		e^{-3x}	-1					
8	$\sqrt{\lambda^{-1}e^{-x}}$	$\sqrt{\lambda^{-1}e^x}$	e^{-3x}		e^{-x}						
9		1	$-\sqrt{\lambda^{-1}e^{-x}}$	$\sqrt{\lambda^{-1}e^x}$							
10			$\sqrt{\lambda^{-1}e^x}$	$\sqrt{\lambda^{-1}e^{-x}}$							
11					1						

The only non-trivial computation to show the existence of a solution, is to show the existence of the 3×3 -block of v . To show this, we apply the following proposition (see 2.11)

Proposition 23.1 *Let $\begin{pmatrix} d_{1,1}^2 & d_{1,2}^2 & d_{1,3}^2 \\ d_{2,1}^2 & d_{2,2}^2 & d_{2,3}^2 \\ d_{3,1}^2 & d_{3,2}^2 & d_{3,3}^2 \end{pmatrix}$ is a doubly stochastic matrix, and put*

$\alpha = d_{1,1}d_{2,1}$, $\beta = d_{1,2}d_{2,2}$ and $\gamma = d_{1,3}d_{2,3}$. Then there exists a unitary $u = (u_{i,j})$ with $|u_{i,j}|^2 = d_{i,j}$, $i, j = 1, 2, 3$ if and only if

$$\alpha^2 + \beta^2 + \gamma^2 - 2\alpha\gamma - 2\beta\gamma - 2\alpha\beta \leq 0 \quad (23.2)$$

In our situation we have

$$\alpha = \lambda^{-2} e^{3x}, \quad \beta = \lambda^{-2} e^x, \quad \gamma = \lambda^{-1} e^{-4x}$$

for which we may substitute

$$\alpha' = \lambda^{-1} e^{2x}, \quad \beta' = \lambda^{-1}, \quad \gamma' = e^{-5x}$$

If we plug these values into the condition (23.2), and reduce as much as possible by the identity (23.1), we find that a solution exists if and only if

$$-4\rho^2 - \rho - 5 \leq 0,$$

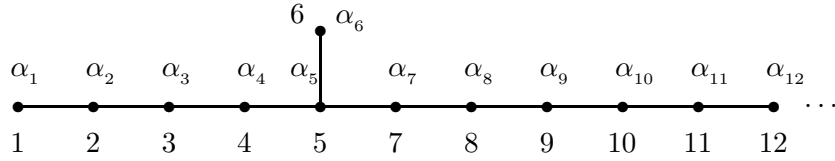
which is clearly satisfied for any positive value of ρ .

By rescaling by complex numbers of modulus 1, we can obtain a solution to the 3×3 -block of v as listed in the table, where ξ, η, ν and μ are complex numbers with modulus 1.

24 A Commuting Square Based on $T(1, 4, \infty)$

24.1 A Not So Successful Attempt

The graph $T(1, 4, \infty)$



If $\lambda = e^x + e^{-x}$ denotes the largest eigenvalue of $T(1, 4, \infty)$, and we by ρ denote e^{2x} , then ρ satisfies the equation (see [Hof])

$$\rho^5 - \rho^3 - \rho^2 - \rho - 1 = 0, \tag{24.1}$$

corresponding to $\lambda \cong 2.04597$.

The coordinates of the corresponding eigenvector are given by

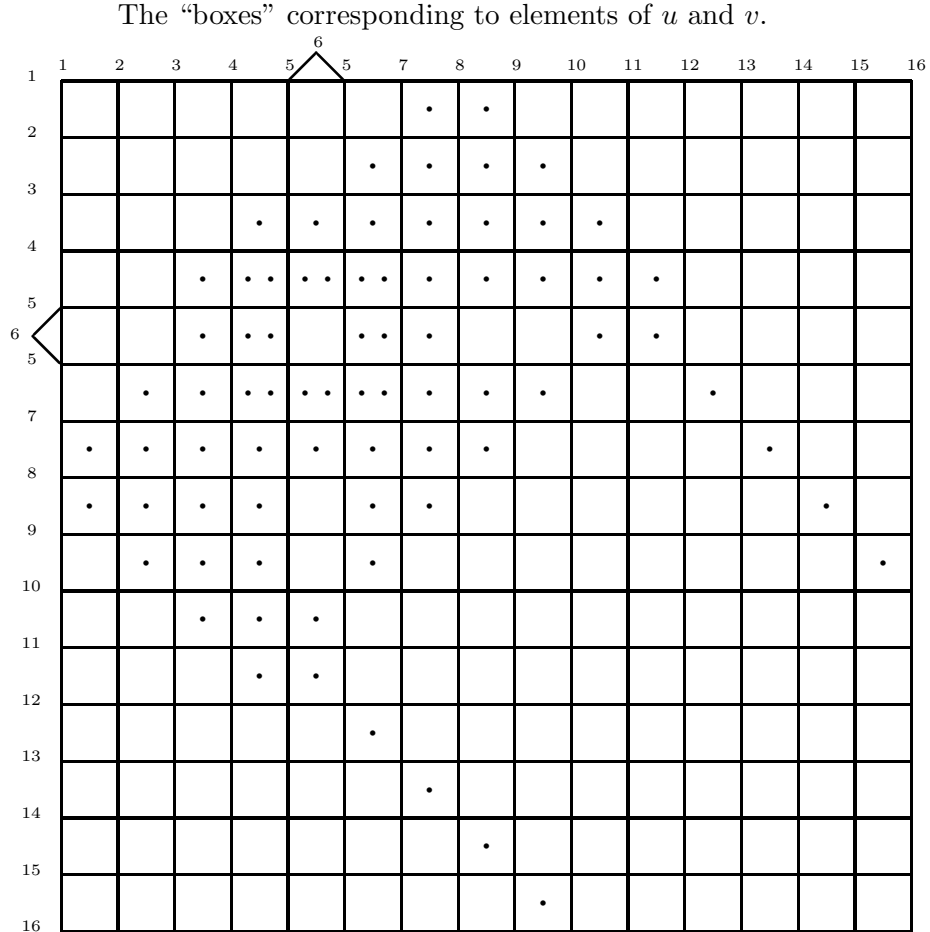
$$\begin{aligned} \alpha_1 &= \frac{\rho^3 - \rho - 1}{\rho \lambda^2} & \alpha_2 &= \frac{\rho^3 - \rho - 1}{\rho \lambda} \\ \alpha_3 &= \rho - 1 & \alpha_4 &= \frac{\rho}{\lambda} \\ \alpha_5 &= 1 & \alpha_6 &= \frac{1}{\lambda} \end{aligned}$$

and for $k \geq 7$ we have $\alpha_k = e^{6-k}$.

In analogy with the examples $T(1,2,\infty)$ and $T(1,3,\infty)$ we will try to construct a commuting square where the non-index defining inclusions are given by a polynomial in the adjacency matrix for $T(1,4,\infty)$. The candidate for the polynomial is, in analogy with the $T(1,2,\infty)$ – and $T(1,3,\infty)$ –cases, given by $S_6(t) = t^6 - 6t^4 + 9t^2 - 2$, and the matrix for $S_6(T(1,4,\infty))$ is given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The entries of the u and v matrices can again be pictured in a diagram, and we have the following “boxes” which define elements of the bi-unitary.



Where the number of dots in a box denotes the dimension of the respective element.

The south-east sloping rows of boxes, starting in the boxes corresponding to the cycles

$\begin{array}{cc} 5 & 7 \\ \square & \\ 12 & 13 \end{array}$ respectively $\begin{array}{cc} 12 & 13 \\ \square & \\ 5 & 7 \end{array}$, all correspond to 1×1 blocks of both u and v .

The scalars defining the transition from u to v are all equal to 1 for these cycles, so a solution of u and v in these boxes is given by putting all the scalars equal to 1.

Using the defined the coordinates of the eigenvector and reducing all expressions involving ρ according to the equation (24.1), we get the following table of the scalars defining the transition from u to v .

The scalars defining the transition from u to v

					6							
	1	2	3	4	5	5	7	8	9	10	11	12
1								$\sqrt{\frac{\rho}{\rho+1}}$	$\sqrt{\frac{1}{\rho+1}}$			
2												
3							$\sqrt{\frac{\rho^2-1}{\rho^3-\rho-1}}$	$\sqrt{\frac{\rho^3-\rho}{\rho^3-\rho-1}}$	$\sqrt{\frac{\rho^2-1}{\rho^3-\rho-1}}$	$\sqrt{\frac{\rho^3-\rho}{\rho^3-\rho-1}}$		
4				$\sqrt{\rho-1}$	$\sqrt{\frac{\rho-1}{\rho}}$	$\sqrt{\frac{\rho^2-1}{\rho^2}}$	$\sqrt{\frac{\rho^2-1}{\rho}}$	$\sqrt{\frac{\rho^2-1}{\rho^2}}$	$\sqrt{\frac{\rho^2-1}{\rho}}$	$\sqrt{\frac{\rho^2-1}{\rho^2}}$	$\sqrt{\frac{\rho^2-1}{\rho^2}}$	
5			$\sqrt{\frac{1}{\rho-1}}$	1	$\sqrt{\frac{1}{\rho}}$	$\sqrt{\frac{\rho+1}{\rho^2}}$	$\sqrt{\frac{\rho+1}{\rho}}$	$\sqrt{\frac{\rho+1}{\rho^2}}$	$\sqrt{\frac{\rho+1}{\rho}}$	$\sqrt{\frac{\rho+1}{\rho^2}}$	$\sqrt{\frac{\rho+1}{\rho}}$	$\sqrt{\frac{\rho+1}{\rho}}$
6			$\sqrt{\frac{\rho}{\rho-1}}$	$\sqrt{\rho}$		$\sqrt{\frac{\rho+1}{\rho}}$	$\sqrt{\rho+1}$			$\sqrt{\frac{\rho+1}{\rho}}$	$\sqrt{\rho+1}$	
7		$\sqrt{\frac{\rho^3-\rho-1}{\rho^2-1}}$	$\sqrt{\frac{\rho^2}{\rho^2-1}}$	$\sqrt{\frac{\rho^2}{\rho+1}}$	$\sqrt{\frac{\rho}{\rho+1}}$	1	$\sqrt{\rho}$	1	$\sqrt{\rho}$			
8	$\sqrt{\frac{\rho+1}{\rho}}$	$\sqrt{\frac{\rho^3-\rho-1}{\rho^3-\rho}}$	$\sqrt{\frac{\rho}{\rho^2-1}}$	$\sqrt{\frac{\rho}{\rho+1}}$	$\sqrt{\frac{1}{\rho+1}}$	$\sqrt{\frac{1}{\rho}}$	1	$\sqrt{\frac{1}{\rho}}$				
9	$\sqrt{\rho+1}$	$\sqrt{\frac{\rho^3-\rho-1}{\rho^2-1}}$	$\sqrt{\frac{\rho^2}{\rho^2-1}}$	$\sqrt{\frac{\rho^2}{\rho+1}}$		1	$\sqrt{\rho}$					
10		$\sqrt{\frac{\rho^3-\rho-1}{\rho^3-\rho}}$	$\sqrt{\frac{\rho}{\rho^2-1}}$	$\sqrt{\frac{\rho}{\rho+1}}$		$\sqrt{\frac{1}{\rho}}$						
11			$\sqrt{\frac{\rho^2}{\rho^2-1}}$	$\sqrt{\frac{\rho^2}{\rho+1}}$	$\sqrt{\frac{\rho}{\rho+1}}$							
12				$\sqrt{\frac{\rho}{\rho+1}}$	$\sqrt{\frac{1}{\rho+1}}$							

The moduli and 2-norms of the elements in u and v can all be determined using the block structure of u and v and the scalars in the previous table. The block structure of u and v is indicated by the thick lines in the diagrams.

The moduli and 2-norms of the entries of u

	1	2	3	4	5	6	5	7	8	9	10	11	12
1									1	1			
2													
3								$\sqrt{\frac{\rho^3-\rho-1}{\rho^2-1}}$	$\sqrt{\frac{\rho^2+\rho-\rho^3}{\rho^2-1}}$	$\sqrt{\frac{1}{\rho^3-\rho}}$	$\sqrt{\frac{\rho^3-\rho-1}{\rho^3-\rho}}$		
4													
5													
6													
7													
8													
9													
10													
11													
12													

The moduli and 2-norms of the entries of v

	1	2	3	4	5	6	5	7	8	9	10	11	12
1								$\sqrt{\frac{\rho}{\rho+1}}$	$\sqrt{\frac{1}{\rho+1}}$				
2							1	$\sqrt{\frac{1}{\rho+1}}$	$\sqrt{\frac{\rho}{\rho+1}}$	1			
3													
4				$\sqrt{\rho-1}$	$\sqrt{\frac{\rho-1}{\rho}}$	$\sqrt{\frac{\rho+1-\rho^2}{\rho}}$	$\sqrt{\frac{\rho^3-\rho-1}{\rho}}$	$\sqrt{\frac{\rho^3-\rho-1}{\rho^3}}$	$\frac{1}{\rho}$	$\sqrt{\frac{\rho^2-1}{\rho^2}}$			
5			1	$\sqrt{2-\rho}$	$\sqrt{\frac{1}{\rho}}$	$\sqrt{\frac{\rho^2-1}{\rho}}$	$\sqrt{\frac{\rho^3-\rho-1}{\rho^3}}$	$\sqrt{\frac{\rho^3-\rho-1}{\rho}}$	$\sqrt{\frac{\rho^2-1}{\rho^2}}$	$\frac{1}{\rho}$	1		
6			1	\vdots	\vdots	\vdots	1				1	1	
7				\vdots	\vdots	\vdots							
8		$\sqrt{\frac{\rho^3-\rho-1}{\rho^2-1}}$	$\sqrt{\frac{\rho^2+\rho-\rho^3}{\rho^2-1}}$	$\sqrt{\frac{\rho^4-1}{\rho^4}}$	$\sqrt{\frac{\rho}{\rho+1}}$	$\sqrt{\frac{\rho^2-1}{\rho^2}}$	$\sqrt{\frac{1}{\rho}}$	$\sqrt{\frac{\rho-1}{\rho}}$	1				
9	1	$\sqrt{\frac{\rho^2+\rho-\rho^3}{\rho^2-1}}$	$\sqrt{\frac{\rho^3-\rho-1}{\rho^2-1}}$	$\frac{1}{\rho^2}$	$\sqrt{\frac{1}{\rho+1}}$	$\frac{1}{\rho}$	$\sqrt{\frac{\rho-1}{\rho}}$	$\sqrt{\frac{1}{\rho}}$					
10	1	$\sqrt{\frac{1}{\rho^3-\rho}}$	$\sqrt{\frac{\rho^3-\rho-1}{\rho^3-\rho}}$	$\sqrt{\frac{1}{\rho}}$		$\sqrt{\frac{\rho-1}{\rho}}$	1						
11		$\sqrt{\frac{\rho^3-\rho-1}{\rho^3-\rho}}$	$\sqrt{\frac{1}{\rho^3-\rho}}$	$\sqrt{\frac{\rho-1}{\rho}}$		$\sqrt{\frac{1}{\rho}}$							
12				$\sqrt{\frac{\rho}{\rho+1}}$	$\sqrt{\frac{1}{\rho+1}}$								

That the squares of the numbers in the blocks form doubly stochastic matrices is easily seen using the equation (24.1). So we need to check whether we can find “phases” which will make the matrices unitary.

The critical part is the existence of vectors with the “right” inner products in the boxes

From u

\ddots $\sqrt{2-\rho}$	\ddots 1	\ddots $\sqrt{\frac{\rho^4-1}{\rho^4}}$
\ddots $\sqrt{\frac{1}{\rho}}$		\ddots $\sqrt{\frac{\rho}{\rho+1}}$
\ddots $\sqrt{\frac{\rho^2-1}{\rho}}$	\ddots 1	\ddots $\sqrt{\frac{\rho^2-1}{\rho^2}}$

From v

\vdots $\sqrt{2-\rho}$	\vdots $\sqrt{\frac{1}{\rho}}$	\vdots $\sqrt{\frac{\rho^2-1}{\rho}}$
\vdots 1		\vdots 1
\vdots $\sqrt{\frac{\rho^4-1}{\rho^4}}$	\vdots $\sqrt{\frac{\rho}{\rho+1}}$	\vdots $\sqrt{\frac{\rho^2-1}{\rho^2}}$

If a solution to these vectors has been found, all the other elements can be determined. This can be seen as follows. The “phases” in this 3×3 part of u and v , influences the other blocks of u and v via the transition from u to v and vice versa. However this influence is restricted to at most one entry of a 2×2 – or 1×1 –block, and so phases for the entire 2×2 – resp. 1×1 –block can be determined to make it unitary. The phases determined in a 2×2 –block may influence other blocks, but again at most one entry in a block is determined this way, and we can continue the argument as above.

If we rescale the first column, of the part coming from u , by $\sqrt{\lambda\alpha_4} = \sqrt{\rho}$, the second by $\sqrt{\lambda\alpha_6} = 1$ and the third by $\sqrt{\lambda\alpha_7} = \sqrt{\lambda e^{-x}}$, and if we also rescale the first row, of the part coming from v , by $\sqrt{\rho}$, the second by 1 and the third by $\sqrt{\lambda e^{-x}}$, the moduli in both blocks becomes

$\sqrt{\rho(2-\rho)}$	1	$\sqrt{\rho^2-1}$
1		1
$\sqrt{\rho^2-1}$	1	$\sqrt{\frac{2+2\rho-\rho^3}{\rho}}$

And hence all we need to find are 9 vectors $(e_{i,j})_{i,j+1}^3$ with

$$\begin{pmatrix} \|e_{1,1}\|_2^2 & \|e_{1,2}\|_2^2 & \|e_{1,3}\|_2^2 \\ \|e_{2,1}\|_2^2 & \|e_{2,2}\|_2^2 & \|e_{2,3}\|_2^2 \\ \|e_{3,1}\|_2^2 & \|e_{3,2}\|_2^2 & \|e_{3,3}\|_2^2 \end{pmatrix} = \begin{pmatrix} \rho(2-\rho) & 1 & \rho^2-1 \\ 1 & 0 & 1 \\ \rho^2-1 & 1 & \frac{2+2\rho-\rho^3}{\rho} \end{pmatrix}$$

and the “right” inner products.

More precisely, having the right inner products can be stated as

$$\sum_i e_{i,j} \otimes \bar{e}_{i,j} = c_j I \text{ and } \sum_j e_{i,j} \otimes \bar{e}_{i,j} = c_i I$$

where

$$\begin{aligned} c_1 &= \frac{1}{2}(\rho(2 - \rho) + 1 + \rho^2 - 1) = \rho \\ c_2 &= 1 \\ c_3 &= \frac{1}{2}(\rho^2 + \frac{2+2\rho-\rho^3}{\rho}) = \frac{\rho+1}{\rho} \end{aligned}$$

Proposition 24.1 *There exists 9 vectors $(e_{i,j})_{i,j+1}^3 \in \mathbb{C}^2$ with*

$$\begin{pmatrix} \|e_{1,1}\|_2^2 & \|e_{1,2}\|_2^2 & \|e_{1,3}\|_2^2 \\ \|e_{2,1}\|_2^2 & \|e_{2,2}\|_2^2 & \|e_{2,3}\|_2^2 \\ \|e_{3,1}\|_2^2 & \|e_{3,2}\|_2^2 & \|e_{3,3}\|_2^2 \end{pmatrix} = \begin{pmatrix} a & d & b \\ d & 0 & d \\ b & d & c \end{pmatrix}$$

and

$$\sum_i e_{i,j} \otimes \bar{e}_{i,j} = c_j I \text{ and } \sum_j e_{i,j} \otimes \bar{e}_{i,j} = c_i I$$

where $c_1 = \frac{1}{2}(a + b + d)$, $c_2 = d$ and $c_3 = \frac{1}{2}(b + c + d)$ if and only if

$$b^2 + d^2 \geq \frac{1}{2}(a^2 + c^2) \text{ and } |b^2 - d^2| \leq ac.$$

Proof Again we consider the map (see lemma 6.10)

$$q : \mathbb{C}^2 \rightarrow \{ \text{self-adjoint } 2 \times 2 \text{ matrices with trace } 0 \} \cong \mathbb{R}^3$$

given by

$$q(x) = \sqrt{2}(x \otimes \bar{x} - \frac{1}{2}\|x\|_2^2 I).$$

Then $\|q(x)\|_{HS} = \|x\|_2^2$ since

$$\|q(x)\|_{HS}^2 = 2(\|x \otimes \bar{x}\|_{HS}^2 - \frac{1}{2}\|x\|_2^4) = \|x\|_2^4.$$

Also $(q(x), q(y)) = 2|(x, y)| - \|x\|_2^2 \|y\|_2^2$, so if $|(x, y)| = \|x\|_2 \|y\|_2 \cos \theta$, $0 \leq \theta \leq 2\pi$ then

$$(q(x), q(y)) = \|x\|_2^2 \|y\|_2^2 (2 \cos \theta - 1) = \|q(x)\| \|q(y)\| \cos 2\theta.$$

Put $q_{i,j} = q(e_{i,j})$ for $i, j = 1, 2, 3$. Then

$$\sum_i q_{i,j} = 0 \text{ and } \sum_j q_{i,j} = 0$$

since the sums are scalars with trace 0.

Hence we have

$$\begin{pmatrix} q_{1,1} & q_{1,2} & q_{1,3} \\ q_{2,1} & q_{2,2} & q_{2,3} \\ q_{3,1} & q_{3,2} & q_{3,3} \end{pmatrix} = \begin{pmatrix} q_{1,1} & q_{1,2} & q_{1,3} \\ q_{2,1} & 0 & -q_{2,1} \\ q_{3,1} & -q_{1,2} & q_{3,3} \end{pmatrix}$$

and get

$$q_{3,3} = -q_{3,1} + q_{1,2} = q_{1,1} + q_{1,2} + q_{2,1}$$

which implies

$$\begin{aligned} (q_{1,1}, q_{2,1}) &= \frac{1}{2}(\|q_{1,1} + q_{2,1}\|_2^2 - \|q_{1,1}\|_2^2 - \|q_{2,1}\|_2^2) \\ &= \frac{1}{2}(\|q_{3,1}\|_2^2 - \|q_{1,1}\|_2^2 - \|q_{2,1}\|_2^2) \\ &= \frac{1}{2}(b^2 - a^2 - d^2). \end{aligned}$$

Similarly we get

$$(q_{1,1}, q_{1,2}) = \frac{1}{2}(b^2 - a^2 - d^2)$$

and we have

$$(q_{1,1}, q_{2,1} + q_{1,2}) = b^2 - a^2 - d^2 \text{ and } (q_{1,1}, q_{2,1} - q_{1,2}) = 0.$$

Also $q_{3,3} = q_{1,1} + (q_{2,1} + q_{1,2})$, hence

$$\|q_{3,3}\|_2^2 = \|q_{1,1}\|_2^2 + \|q_{2,1} + q_{1,2}\|_2^2 + 2(q_{1,1}, q_{2,1} + q_{1,2}),$$

and we get

$$\begin{aligned} \|q_{2,1} + q_{1,2}\|_2^2 &= c^2 - a^2 - 2(b^2 - a^2 - d^2) \\ &= a^2 + c^2 - 2b^2 + 2d^2 \end{aligned} \tag{24.2}$$

and

$$\begin{aligned} \|q_{2,1} - q_{1,2}\|_2^2 &= 2\|q_{2,1}\|_2^2 + 2\|q_{1,2}\|_2^2 - \|q_{2,1} + q_{1,2}\|_2^2 \\ &= 4d^2 - a^2 - c^2 + 2b^2 + 2d^2 \\ &= 2d^2 + 2b^2 - a^2 - c^2. \end{aligned} \tag{24.3}$$

This now implies

$$\|q_{3,3} - q_{1,1}\|_2^2 = \|q_{2,1} + q_{1,2}\|_2^2 = a^2 + c^2 - 2b^2 + 2d^2$$

and hence

$$(q_{1,1}, q_{3,3}) = \frac{1}{2}(\|q_{1,1}\|_2^2 + \|q_{3,3}\|_2^2 - \|q_{1,1} - q_{3,3}\|_2^2) = b^2 - d^2.$$

Cauchy-Schwartz gives

$$|b^2 - d^2| \leq \|q_{1,1}\|_2 \|q_{3,3}\|_2 = ac.$$

From (24.2) and (24.3) we must have

$$2d^2 + 2b^2 \geq a^2 + c^2.$$

To show that the condition is also sufficient we argue as follows.

Assume the two conditions are satisfied. Choose $q_{1,1}$ and $q_{3,3}$ in \mathbb{C}^3 with the right length and inner product, and pick $h \in \mathbb{R}^3$ with $(h, q_{1,1}) = 0 = (h, q_{3,3})$ and $\|h\|_2^2 = 2d^2 + 2b^2 - a^2 - c^2$. Put

$$q_{2,1} = \frac{1}{2}(q_{3,3} - q_{1,1} + h) \text{ and } q_{1,2} = \frac{1}{2}(q_{3,3} - q_{1,1} - h).$$

Put also

$$q_{1,3} = -q_{1,1} - q_{2,1} \text{ and } q_{3,1} = -q_{1,1} - q_{1,2}$$

Then the defined vectors have the right length and inner products.

Now identify \mathbb{R}^3 with the 2×2 matrices with trace 0, and pick $e_{i,j}$ such that

$$\begin{aligned} q(e_{1,1}) &= q_{1,1} & , & & q(e_{1,2}) &= q_{1,2} \\ q(e_{1,3}) &= q_{1,3} & , & & q(e_{2,1}) &= q_{2,1} \\ q(e_{2,3}) &= q_{2,3} & , & & q(e_{3,1}) &= q_{3,1} \\ q(e_{3,2}) &= q_{3,2} & , & & q(e_{3,3}) &= q_{3,3} \end{aligned}$$

□

If we apply the above proposition to our construction for $S_6(T(1, 4, \infty))$, we have $a = \rho(2 - \rho)$,

$b = \rho^2 - 1$, $c = \frac{(\rho^2-1)(\rho+1)}{\rho^3}$ and $d = 1$. In numerical entities we have $a \cong 0.71468$,

$b \cong 1.35364$ and $c \cong 0.95001$ and we see that the first criterion of proposition 24.1 is satisfied, but $|b^2 - d^2| \cong 0.83234$ and $ac \cong 0.67895$, so the second condition is not satisfied.

24.2 A Successful Attempt

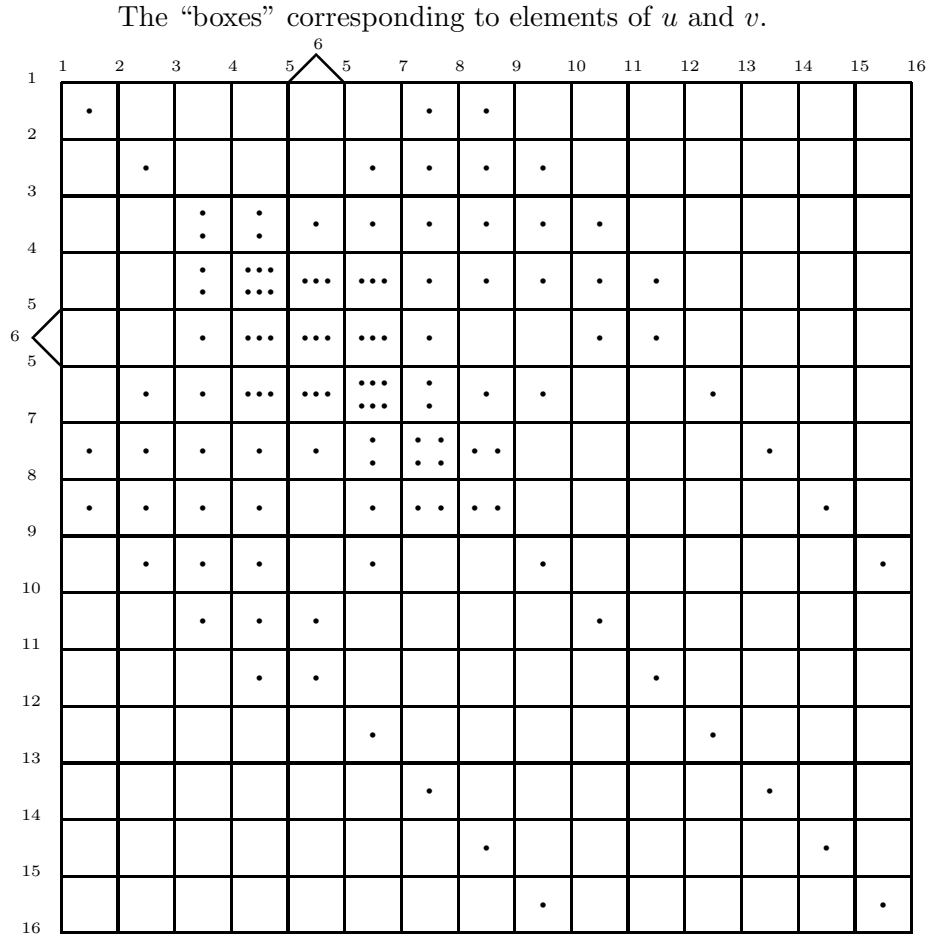
Since the first attempt to construct a commuting square with $T(1, 3, \infty)$ as the index defining side, did not succeed, we did a little experimenting and found another polynomial which will do the job.

We shall be looking at the polynomial $S_6(t) + 1 = t^6 - 6t^4 + 9t^2 - 1$.

$(S_6 + 1)(T(1, 4, \infty))$ is given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 3 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 2 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

and the boxes corresponding to entries of u or v are given as



Where the number of dots in a box denotes the dimension of the respective element.

The south-east sloping rows of boxes, starting in the boxes corresponding to the cycles

$$\begin{array}{c} 5 \\ \square \\ 12 \end{array} \begin{array}{c} 7 \\ \square \\ 13 \end{array}, \quad \begin{array}{c} 12 \\ \square \\ 5 \end{array} \begin{array}{c} 13 \\ \square \\ 7 \end{array} \quad \text{respectively} \quad \begin{array}{c} 10 \\ \square \\ 10 \end{array} \begin{array}{c} 9 \\ \square \\ 9 \end{array}, \quad \text{all correspond to } 1 \times 1 \text{ blocks of both } u \text{ and } v.$$

The scalars defining the transition from u to v are all equal to 1 for these cycles, so a solution of u and v in these boxes, is given by putting all the scalars equal to 1. Again the problem of finding a solution to u and v , is reduced to finding a solution in the upper left corner of the diagram. Here we look at

The scalars defining the transition from u to v

					6								
	1	2	3	4	5	5	7	8	9	10	11	12	
1	1						$\sqrt{\frac{\rho}{\rho+1}}$	$\sqrt{\frac{1}{\rho+1}}$					
2													
3		1				$\sqrt{\frac{\rho^2-1}{\rho^3-\rho-1}}$	$\sqrt{\frac{\rho^3-\rho}{\rho^3-\rho-1}}$	$\sqrt{\frac{\rho^2-1}{\rho^3-\rho-1}}$	$\sqrt{\frac{\rho^3-\rho}{\rho^3-\rho-1}}$				
4			1	$\sqrt{\rho-1}$	$\sqrt{\frac{\rho-1}{\rho}}$	$\sqrt{\frac{\rho^2-1}{\rho^2}}$	$\sqrt{\frac{\rho^2-1}{\rho}}$	$\sqrt{\frac{\rho^2-1}{\rho^2}}$	$\sqrt{\frac{\rho^2-1}{\rho}}$	$\sqrt{\frac{\rho^2-1}{\rho^2}}$			
5			$\sqrt{\frac{1}{\rho-1}}$	1	$\sqrt{\frac{1}{\rho}}$	$\sqrt{\frac{\rho+1}{\rho^2}}$	$\sqrt{\frac{\rho+1}{\rho}}$	$\sqrt{\frac{\rho+1}{\rho^2}}$	$\sqrt{\frac{\rho+1}{\rho}}$	$\sqrt{\frac{\rho+1}{\rho^2}}$	$\sqrt{\frac{\rho+1}{\rho}}$		
6			$\sqrt{\frac{\rho}{\rho-1}}$	$\sqrt{\rho}$	1	$\sqrt{\frac{\rho+1}{\rho}}$	$\sqrt{\rho+1}$			$\sqrt{\frac{\rho+1}{\rho}}$	$\sqrt{\rho+1}$		
7		$\sqrt{\frac{\rho^3-\rho-1}{\rho^2-1}}$	$\sqrt{\frac{\rho^2}{\rho^2-1}}$	$\sqrt{\frac{\rho^2}{\rho+1}}$	$\sqrt{\frac{\rho}{\rho+1}}$	1	$\sqrt{\rho}$	1	$\sqrt{\rho}$				
8	$\sqrt{\frac{\rho+1}{\rho}}$	$\sqrt{\frac{\rho^3-\rho-1}{\rho^3-\rho}}$	$\sqrt{\frac{\rho}{\rho^2-1}}$	$\sqrt{\frac{\rho}{\rho+1}}$	$\sqrt{\frac{1}{\rho+1}}$	$\sqrt{\frac{1}{\rho}}$	1	$\sqrt{\frac{1}{\rho}}$					
9	$\sqrt{\rho+1}$	$\sqrt{\frac{\rho^3-\rho-1}{\rho^2-1}}$	$\sqrt{\frac{\rho^2}{\rho^2-1}}$	$\sqrt{\frac{\rho^2}{\rho+1}}$		1	$\sqrt{\rho}$	1					
10		$\sqrt{\frac{\rho^3-\rho-1}{\rho^3-\rho}}$	$\sqrt{\frac{\rho}{\rho^2-1}}$	$\sqrt{\frac{\rho}{\rho+1}}$		$\sqrt{\frac{1}{\rho}}$			1				
11			$\sqrt{\frac{\rho^2}{\rho^2-1}}$	$\sqrt{\frac{\rho^2}{\rho+1}}$	$\sqrt{\frac{\rho}{\rho+1}}$					1			
12				$\sqrt{\frac{\rho}{\rho+1}}$	$\sqrt{\frac{1}{\rho+1}}$							1	

Again the moduli, 2-norms and Hilbert-Schmidt norms of the elements in u and v can be determined using the block structure of u and v and the scalars in the above table. The block structure of u and v is indicated by the thick lines in the diagrams.

The moduli, 2-norms and Hilbert-Schmidt norms of the entries of u

	1	2	3	4	5	6	5	7	8	9	10	11	12
1	1							1	1				
2		1					$\sqrt{\frac{\rho^3-\rho-1}{\rho^2-1}}$	$\sqrt{\frac{\rho^2+\rho-\rho^3}{\rho^2-1}}$	$\sqrt{\frac{1}{\rho^3-\rho}}$	$\sqrt{\frac{\rho^3-\rho-1}{\rho^3-\rho}}$			
3			\vdots	\vdots			$\sqrt{\frac{\rho+1-\rho^2}{\rho-1}}$	$\sqrt{\frac{\rho^3-\rho-1}{\rho^2-1}}$	$\sqrt{\frac{\rho^3-\rho-1}{\rho^3-\rho}}$	$\sqrt{\frac{1}{\rho^3-\rho}}$		1	
4			\vdots	\vdots	\vdots		$\sqrt{\frac{\rho^4-1}{\rho^4}}$	$\frac{1}{\rho^2}$	$\sqrt{\frac{1}{\rho}}$	$\sqrt{\frac{\rho-1}{\rho}}$	$\sqrt{\frac{1}{\rho+1}}$	$\sqrt{\frac{\rho}{\rho+1}}$	
5			$\sqrt{\frac{\rho-1}{\rho}}$	$\sqrt{\frac{1}{\rho}}$	1		$\sqrt{\frac{\rho}{\rho+1}}$	$\sqrt{\frac{1}{\rho+1}}$			$\sqrt{\frac{\rho}{\rho+1}}$	$\sqrt{\frac{1}{\rho+1}}$	
6			$\sqrt{\frac{\rho-1}{\rho}}$	$\sqrt{\frac{1}{\rho}}$	1		$\sqrt{\frac{\rho}{\rho+1}}$	$\sqrt{\frac{1}{\rho+1}}$			$\sqrt{\frac{\rho}{\rho+1}}$	$\sqrt{\frac{1}{\rho+1}}$	
7		1	$\sqrt{\frac{\rho+1-\rho^2}{\rho}}$	$\sqrt{\frac{\rho^2-1}{\rho}}$	1		$\sqrt{\frac{2\rho^2-1}{\rho^2}}$	$\frac{1}{\rho}$	$\sqrt{\frac{\rho-1}{\rho}}$	$\sqrt{\frac{1}{\rho}}$			
8	$\sqrt{\frac{\rho}{\rho+1}}$	$\sqrt{\frac{1}{\rho+1}}$	$\sqrt{\frac{\rho+1}{\rho^3}}$	$\sqrt{\frac{\rho^3-\rho-1}{\rho^3}}$	1	\vdots	$\sqrt{\frac{1}{\rho}}$	$\sqrt{\frac{2\rho-1}{\rho}}$	1				
9	$\sqrt{\frac{1}{\rho+1}}$	$\sqrt{\frac{\rho}{\rho+1}}$	$\sqrt{\frac{\rho^3-\rho-1}{\rho^3}}$	$\sqrt{\frac{\rho+1}{\rho^3}}$		\vdots	$\sqrt{\frac{\rho-1}{\rho}}$	$\sqrt{\frac{1}{\rho}}$	1				
10		1	$\frac{1}{\rho}$	$\sqrt{\frac{\rho^2-1}{\rho^2}}$			1			1			
11			$\sqrt{\frac{\rho^2-1}{\rho^2}}$	$\frac{1}{\rho}$	1						1		
12				1	1								1

The moduli, 2-norms and Hilbert-Schmidt norms of the entries of v

					6							
	1	2	3	4	5	5	7	8	9	10	11	12
1	1						$\sqrt{\frac{\rho}{\rho+1}}$	$\sqrt{\frac{1}{\rho+1}}$				
2		1				1	$\sqrt{\frac{1}{\rho+1}}$	$\sqrt{\frac{\rho}{\rho+1}}$	1			
3										
4			1	$\sqrt{\rho-1}$	$\sqrt{\frac{\rho-1}{\rho}}$	$\sqrt{\frac{\rho+1-\rho^2}{\rho}}$	$\sqrt{\frac{\rho^3-\rho-1}{\rho}}$	$\sqrt{\frac{\rho^3-\rho-1}{\rho^3}}$	$\frac{1}{\rho}$	$\sqrt{\frac{\rho^2-1}{\rho^2}}$		
4			..	⋮	⋮	⋮						
5			1	$\sqrt{3-\rho}$	$\sqrt{\frac{1}{\rho}}$	$\sqrt{\frac{\rho^2-1}{\rho}}$	$\sqrt{\frac{\rho^3-\rho-1}{\rho^3}}$	$\sqrt{\frac{\rho^3-\rho-1}{\rho}}$	$\sqrt{\frac{\rho^2-1}{\rho^2}}$	$\frac{1}{\rho}$	1	
6			1	⋮	⋮	⋮						
5			1	1	1	1	1			1	1	
5				⋮	⋮	⋮	..					
7		$\sqrt{\frac{\rho^3-\rho-1}{\rho^2-1}}$	$\sqrt{\frac{\rho^2+\rho-\rho^3}{\rho^2-1}}$	$\sqrt{\frac{\rho^4-1}{\rho^4}}$	$\sqrt{\frac{\rho}{\rho+1}}$	$\sqrt{\frac{2\rho^2-1}{\rho^2}}$	$\sqrt{\frac{1}{\rho}}$	$\sqrt{\frac{\rho-1}{\rho}}$	1			
8	1	$\sqrt{\frac{\rho^2+\rho-\rho^3}{\rho^2-1}}$	$\sqrt{\frac{\rho^3-\rho-1}{\rho^2-1}}$	$\frac{1}{\rho^2}$	$\sqrt{\frac{1}{\rho+1}}$	$\frac{1}{\rho}$	$\sqrt{\frac{2\rho-1}{\rho}}$	$\sqrt{\frac{1}{\rho}}$				
9	1	$\sqrt{\frac{1}{\rho^3-\rho}}$	$\sqrt{\frac{\rho^3-\rho-1}{\rho^3-\rho}}$	$\sqrt{\frac{1}{\rho}}$		$\sqrt{\frac{\rho-1}{\rho}}$	1	1				
10		$\sqrt{\frac{\rho^3-\rho-1}{\rho^3-\rho}}$	$\sqrt{\frac{1}{\rho^3-\rho}}$	$\sqrt{\frac{\rho-1}{\rho}}$		$\sqrt{\frac{1}{\rho}}$			1			
11			1	$\sqrt{\frac{1}{\rho+1}}$	$\sqrt{\frac{\rho}{\rho+1}}$						1	
12				$\sqrt{\frac{\rho}{\rho+1}}$	$\sqrt{\frac{1}{\rho+1}}$							1

We will first show that there is a solution to the parts of u and v involving

From u

$\begin{smallmatrix} \vdots \\ \vdots \\ \vdots \end{smallmatrix}$ $\sqrt{3-\rho}$	$\begin{smallmatrix} \cdots \\ \cdots \\ \cdots \end{smallmatrix}$ 1	$\begin{smallmatrix} \cdots \\ \cdots \\ \cdots \end{smallmatrix}$ $\sqrt{\frac{\rho^4-1}{\rho^4}}$
$\begin{smallmatrix} \cdots \\ \cdots \\ \cdots \end{smallmatrix}$ $\sqrt{\frac{1}{\rho}}$	$\begin{smallmatrix} \cdots \\ \cdots \\ \cdots \end{smallmatrix}$ 1	$\begin{smallmatrix} \cdots \\ \cdots \\ \cdots \end{smallmatrix}$ $\sqrt{\frac{\rho}{\rho+1}}$
$\begin{smallmatrix} \cdots \\ \cdots \\ \cdots \end{smallmatrix}$ $\sqrt{\frac{\rho^2-1}{\rho}}$	$\begin{smallmatrix} \cdots \\ \cdots \\ \cdots \end{smallmatrix}$ 1	$\begin{smallmatrix} \vdots \\ \vdots \\ \vdots \end{smallmatrix}$ $\sqrt{\frac{2\rho^2-1}{\rho^2}}$

From v

$\begin{smallmatrix} \vdots \\ \vdots \\ \vdots \end{smallmatrix}$ $\sqrt{3-\rho}$	$\begin{smallmatrix} \vdots \\ \vdots \\ \vdots \end{smallmatrix}$ $\sqrt{\frac{1}{\rho}}$	$\begin{smallmatrix} \vdots \\ \vdots \\ \vdots \end{smallmatrix}$ $\sqrt{\frac{\rho^2-1}{\rho}}$
$\begin{smallmatrix} \vdots \\ \vdots \\ \vdots \end{smallmatrix}$ 1	$\begin{smallmatrix} \vdots \\ \vdots \\ \vdots \end{smallmatrix}$ 1	$\begin{smallmatrix} \vdots \\ \vdots \\ \vdots \end{smallmatrix}$ 1
$\begin{smallmatrix} \vdots \\ \vdots \\ \vdots \end{smallmatrix}$ $\sqrt{\frac{\rho^4-1}{\rho^4}}$	$\begin{smallmatrix} \vdots \\ \vdots \\ \vdots \end{smallmatrix}$ $\sqrt{\frac{\rho}{\rho+1}}$	$\begin{smallmatrix} \vdots \\ \vdots \\ \vdots \end{smallmatrix}$ $\sqrt{\frac{2\rho^2-1}{\rho^2}}$

If we rescale, as in the previous computation, the moduli and norms of the involved entities become

$\begin{smallmatrix} \vdots \\ \vdots \\ \vdots \end{smallmatrix}$ $\sqrt{\rho(3-\rho)}$	$\begin{smallmatrix} \cdots \\ \cdots \\ \cdots \end{smallmatrix}$ 1	$\begin{smallmatrix} \cdots \\ \cdots \\ \cdots \end{smallmatrix}$ $\sqrt{\rho^2-1}$
$\begin{smallmatrix} \cdots \\ \cdots \\ \cdots \end{smallmatrix}$ 1	$\begin{smallmatrix} \cdots \\ \cdots \\ \cdots \end{smallmatrix}$ 1	$\begin{smallmatrix} \cdots \\ \cdots \\ \cdots \end{smallmatrix}$ 1
$\begin{smallmatrix} \cdots \\ \cdots \\ \cdots \end{smallmatrix}$ $\sqrt{\rho^2-1}$	$\begin{smallmatrix} \cdots \\ \cdots \\ \cdots \end{smallmatrix}$ 1	$\begin{smallmatrix} \vdots \\ \vdots \\ \vdots \end{smallmatrix}$ $\sqrt{\frac{3+3\rho-\rho^3}{\rho}}$

Put $a = \rho(3-\rho) \cong 2.249$, $b = \rho^2-1 \cong 1.354$, $c = \frac{1}{\rho}(3+3\rho-\rho^3) \cong 2.602$ and $d = 1$. Then if we look for a symmetric solution for u , the left-hand and right-hand columns above are part of matrices of the form

$$\begin{pmatrix} (2 \times 1)_1 & (2 \times 3)_1 \\ \xi & e \\ \eta & f \end{pmatrix} \quad \begin{pmatrix} f & \nu \\ h & \mu \\ (2 \times 3)_2 & (2 \times 1)_2 \end{pmatrix} \quad (24.4)$$

where $e, f, h \in \mathbb{C}^3$ with $\|e\|_2^2 = d$, $\|f\|_2^2 = b$, $\|h\|_2^2 = d$ and $\xi, \eta, \nu, \mu \in \mathbb{C}$. The left-hand matrix is of the form $\sqrt{\frac{a+b+d}{3}}$ unitary and the right-hand matrix is of the form $\sqrt{\frac{c+b+d}{3}}$ unitary.

If we can find e, f, h, ξ, η, ν and μ with the desired properties, the rest of the above matrices can be determined by extending to an orthonormal basis of \mathbb{C}^4 in either case.

Since a rescaling of the vectors e and h gives columns in a unitary in v , we have $e \perp h$.

We also have

$$\begin{aligned} |\xi|^2 &= \frac{a+b+d}{3} - d = \frac{a+b-2d}{3} \geq 0 & \text{if } a \geq b-2d \\ |\eta|^2 &= \frac{a+b+d}{3} - b = \frac{a-2b+d}{3} \geq 0 & \text{if } a \geq d-2b \end{aligned}.$$

The criteria for positivity are both satisfied. Hence

$$|(e, f)| = |\xi\eta| = \frac{1}{3}\sqrt{(a+b-2d)(a-2b+d)} = \frac{1}{3}\sqrt{-2(b-d)^2 + (a-b)(a-d)}.$$

Similarly we get

$$\begin{aligned} |\nu|^2 &= \frac{c+b+d}{3} - d = \frac{c+b-2d}{3} \geq 0 & \text{if } c \geq b-2d \\ |\mu|^2 &= \frac{c+b+d}{3} - b = \frac{c-2b+d}{3} \geq 0 & \text{if } c \geq d-2b. \end{aligned}$$

Again both criteria for positivity are satisfied, and we get

$$|(f, h)| = |\mu\nu| = \frac{1}{3}\sqrt{(c+b-2d)(c-2b+d)} = \frac{1}{3}\sqrt{-2(b-d)^2 + (c-b)(c-d)}.$$

In particular, a necessary condition for a solution to the bi-unitary problem is

$$a \geq \max\{b-2d, d-2b\} \quad \text{and} \quad c \geq \max\{b-2d, d-2b\},$$

which is seen to be satisfied by the values of a , b , c , and d above.

Let k_1 , k_2 and k_3 be an orthonormal basis for \mathbb{C}^3 , and put

$$e = dk_1, \quad h = dk_2 \quad \text{and} \quad f = \gamma_1 k_1 + \gamma_2 k_2 + \gamma_3 k_3,$$

where $\gamma_1 = \frac{1}{3}\sqrt{-2(b-d)^2 + (a-b)(a-d)}$ and $\gamma_2 = \frac{1}{3}\sqrt{-2(b-d)^2 + (c-b)(c-d)}$.

If γ_3 can be chosen such that $|\gamma_1|^2 + |\gamma_2|^2 + |\gamma_3|^2 = b$ then

$$\begin{aligned} \|e\|^2 &= \|h\|^2 = d, \quad \|f\|^2 = b, \quad (e, h) = 0, \\ (e, f) &= \frac{1}{3}\sqrt{-2(b-d)^2 + (a-b)(a-d)} \quad \text{and} \quad (h, f) = \frac{1}{3}\sqrt{-2(b-d)^2 + (c-b)(c-d)}. \end{aligned}$$

Computing, we get

$$b - |\gamma_1|^2 - |\gamma_2|^2 = \frac{1}{9}(9b - (a-b)(a-d) - (c-b)(c-d) + 4(b-d)^2) \cong 1.031 > 0,$$

so $\gamma_3 = \frac{1}{3}\sqrt{9b - (a-b)(a-d) - (c-b)(c-d) + 4(b-d)^2}$ will do the job.

Since the above solution to e , f and h is real, we can obtain a solution to the part of u corresponding to (24.4) as follows. Put

$$\begin{aligned} \mu' &= \sqrt{\frac{\rho}{3(\rho+1)}}\mu, \quad \nu' = \sqrt{\frac{\rho}{3(\rho+1)}}\nu, \quad \eta' = \frac{1}{\sqrt{3\rho}}\eta, \\ \xi' &= \frac{1}{\sqrt{3\rho}}\xi, \quad \delta_i = \frac{1}{\sqrt{3\rho}}\gamma_i, \quad \epsilon_i = \sqrt{\frac{\rho}{3(\rho+1)}}\gamma_i. \end{aligned}$$

Then, with A_1, A_2, B_1 and B_2 obtained by extending to orthonormal bases of \mathbb{C}^3 , we have the following solution to (24.4)

$$\begin{pmatrix} A_1 & B_1 \\ \xi' & (\frac{1}{\sqrt{3\rho}}, 0, 0) \\ \eta' & (\delta_1, \delta_2, \delta_3) \end{pmatrix} \begin{pmatrix} (\epsilon_1, \epsilon_2, \epsilon_3) & \nu' \\ (\sqrt{\frac{\rho}{3(\rho+1)}}, 0, 0) & \mu' \\ B_2 & A_2 \end{pmatrix}.$$

Extend the above solution to a larger part of u , by reflecting it in the main diagonal of the diagram and then rescale to get the right Hilbert–Schmidt norms. Then the “directions” of the corresponding summands of v is given by:

$$\begin{pmatrix} A_1^t & \xi & \eta \\ B_1^t & e^t & f^t \end{pmatrix} \quad \begin{pmatrix} f^t & h^t & B_2^t \\ \nu & \mu & A_2^t \end{pmatrix},$$

and hence, with the right scaling, unitary.

To show that there is a solution to the rest of u and v we argue as on page 169.

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